Apply Mathematical Techniques in a Manufacturing, Engineering or Related Environment

MEM30012A

Learner’s Guide
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Introduction

Welcome to this unit. This unit applies the concepts of mathematics to appropriate and simple engineering situations within various areas of engineering expertise. The skills you will learn will be also used in more advanced mathematical subjects so it is of great importance that you become proficient with all the techniques covered.

The time needed to complete this unit is 60 nominal hours. Some students will require the full 60 hours whereas others may have existing skills which will enable them to complete the unit sooner.

Prerequisites

Although there are no formal prerequisites, students should have a sound knowledge of lower high school Mathematics before commencing this unit.

How to use this guide

This Learner’s Guide is divided into seven Sections, covering the seven elements of competence.

Each Section is divided into a number of smaller parts or ‘sections’, each of which ends with an activity for you to try. The answers to these activities are provided at the end of the Section you are working on.

To make full use of this guide, carefully read and study the notes and worked examples for a topic and then complete the activity.

Check your answers and, provided that you have made no errors or only minor errors, move on to the next topic.

If you are not happy with your results on a particular activity, reread the notes and work through the examples again. Reading another text may help. If necessary, contact your facilitator for advice.

When you have finished all the topics in a Section, revise your work and then complete the assignment for the Section. If you study this unit by correspondence, send this assignment to your facilitator and then commence the next Section.

Helpful tips

Make sure you understand all the material in this guide along with any other texts that you are using. Please consider the following if you are unsure about the content.

- Talk to or make contact with your facilitator.
- Research books, videos and websites.
- Talk to other students.
Resources

Texts
There is no set text for this unit. This Learner’s Guide incorporates all instructional material relating to the unit.

Internet
There are many mathematics-related resources online. Some of the sites that may be useful for this unit are:
- http://www.mathcentre.ac.uk/students.php

Calculator
Although some parts of the unit can be done without a calculator (the geometry Section, for example), you will need a scientific calculator for most of the other Sections.

The use of a graphics calculator is not recommended. Should you have a graphics calculator, use it for checking answers. In tests and assignments you will always be required to show full working out in order to obtain full marks.
## Elements of competence

<table>
<thead>
<tr>
<th>Element</th>
<th>Performance Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Use concepts of arithmetic in the solution of engineering problems</td>
<td>Units of physical quantities are converted to facilitate engineering calculations. Calculations are performed to solve problems involving rational and irrational numbers. Scientific notation is used to represent numbers. Calculations are checked for reasonableness using estimating and approximating techniques.</td>
</tr>
<tr>
<td>2 Solve engineering problems involving algebraic expressions with one independent variable</td>
<td>Algebraic expressions are manipulated using mathematical operations in their <em>correct order</em>.</td>
</tr>
<tr>
<td>3 Use two-dimensional geometry to solve practical problems</td>
<td>Angles expressed in degrees are correctly converted to radians and vice versa. The perimeter, area, length and angles of a range of two-dimensional figures are correctly calculated. The volume and surface area of complex figures are correctly calculated. Points identified in terms of cartesian coordinates can be converted to polar coordinates and vice versa.</td>
</tr>
<tr>
<td>4 Use trigonometry to solve practical problems</td>
<td>Basic trigonometry functions are used to calculate the lengths of the sides of right triangles. Inverse trigonometry functions are used to determine angles in a right triangle given the lengths of two sides. The sine rule is used to determine the lengths of the sides of acute and obtuse triangles given one side and two angles. The cosine rule is used to determine the lengths of the sides of acute and obtuse triangles given two sides and one angle.</td>
</tr>
<tr>
<td>5 Graph linear functions</td>
<td>Linear functions are solved graphically and equations of straight lines are determined from the slope and one point, or two points. Two linear functions are solved simultaneously both algebraically and geometrically. The length and mid point of a line segment are determined.</td>
</tr>
<tr>
<td>6 Solve quadratic equations</td>
<td>Quadratic equations are solved. Simultaneous linear and quadratic equations are solved.</td>
</tr>
<tr>
<td>7 Perform basic statistical calculations</td>
<td>Mean, median and mode are calculated from given data. Standard deviation is calculated and interpreted employing graphical representation.</td>
</tr>
</tbody>
</table>
Assessment

The Elements and Performance Criteria for this unit are assessed by means of seven assessment tasks, one for each Section. These tasks are assignments consisting of a set of questions.

If you study this unit by correspondence, you should send the assignments to your tutor. If you study this unit in a conventional classroom setting, your facilitator will inform you how this unit will be assessed. This may be by means of the assignments or by means of a series of tests and/or projects.
Section 1 – Arithmetic

1.1 Metric conversion

In all aspects of science and technology, we must be able to measure physical quantities. The concept of ratio is basic to the process of measurement. In fact, to measure a quantity means to determine the ratio of that quantity to some chosen unit. For example, to measure the length of a cable with a metre stick, we determine how many times the length of the metre stick is contained in the length of the cable. If the metre stick can be fitted along the length of the cable six times, the measurement is \( \frac{6}{1} = 6 \) metres. In this example, the unit of measurement is the metre.

By selecting units for some fundamental physical quantities, we form a system of units. In science, technology, and industry, the universally accepted system of units is a metric system called the International System of Units, abbreviated SI (for Système International). The seven SI basic units are listed in the following table.

<table>
<thead>
<tr>
<th>Table 1.1: SI basic units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity</td>
</tr>
<tr>
<td>Length</td>
</tr>
<tr>
<td>Mass</td>
</tr>
<tr>
<td>Time</td>
</tr>
<tr>
<td>Electric current</td>
</tr>
<tr>
<td>Thermodynamic temperature</td>
</tr>
<tr>
<td>Amount of substance</td>
</tr>
<tr>
<td>Luminous intensity</td>
</tr>
</tbody>
</table>

In expressing multiples or sub-multiples of the basic units we use prefixes as listed in the following table.

<table>
<thead>
<tr>
<th>Table 1.2: SI prefixes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
</tr>
<tr>
<td>10^{18}</td>
</tr>
<tr>
<td>10^{15}</td>
</tr>
<tr>
<td>10^{12}</td>
</tr>
<tr>
<td>10^9</td>
</tr>
<tr>
<td>10^8</td>
</tr>
<tr>
<td>10^3</td>
</tr>
<tr>
<td>10^2</td>
</tr>
<tr>
<td>10^1</td>
</tr>
</tbody>
</table>

These prefixes are especially useful because they employ a decimal (power-of-ten) system.
To illustrate, note the factors represented by the prefixes centi, milli and kilo.

\[
\begin{align*}
\text{centi} & \rightarrow \frac{1}{10^2} = \frac{1}{100} \\
\text{milli} & \rightarrow \frac{1}{10^3} = \frac{1}{1000} \\
\text{kilo} & \rightarrow 10^3 = 1000
\end{align*}
\]

In practice, we shall generally use the prefixes from milli through to kilo, but occasionally we may go beyond this range.

Using the metre as our basic unit (we could use any unit, however), we have:

\[
\begin{align*}
1 \text{ centimetre} &= \frac{1}{100} (1 \text{ metre}) \quad \text{or} \quad 1 \text{ cm} = \frac{1}{100} m = \frac{1}{10^2} m \\
1 \text{ millimetre} &= \frac{1}{1000} (1 \text{ metre}) \quad \text{or} \quad 1 \text{ mm} = \frac{1}{1000} m = \frac{1}{10^3} m \\
1 \text{ kilometre} &= 1000 (1 \text{ metre}) \quad \text{or} \quad 1 \text{ km} = 1000 m = 10^3 m
\end{align*}
\]

Another way of stating these relationships is:

\[
1 \text{m }= 100 \text{ cm }= 1000 \text{ mm }= \frac{1}{1000} \text{ km} \quad \text{or} \quad 1 \text{m }= 10^2 \text{ cm }= 10^3 \text{ mm }= \frac{1}{10^3} \text{ km}
\]

**Example 1** Convert 3.5 g to kilograms.

\[
3.5 \div 1000 = 0.0035 \text{ kg}
\]

**Example 2** Convert 2 A to milliamperes.

\[
2 \times 10^3 = 2000 \text{ mA}
\]

**Example 3** Convert 5 km to centimetres.

\[
5 \times 10^3 = 5000 \text{ m}
\]

\[
5000 \times 10^2 = 500 000 \text{ cm}
\]

Special care needs to be taken when converting square (for area) or cubic measurements (for volume).
Consider these examples.

**Example 4**  Convert 2 cm\(^2\) to mm\(^2\)

\[
2 \text{ cm}^2 = 2 \times 1 \text{ cm} \times 1 \text{ cm} \
= 2 \times 10 \text{ mm} \times 10 \text{ mm} \
= 200 \text{ mm}^2
\]

**Example 5**  Convert 2 300 m\(^3\) to km\(^3\).

\[
2300 \text{ m}^3 = 2300 \times 1 \text{ m} \times 1 \text{ m} \times 1 \text{ m} \
= 2300 \times \frac{1}{1000} \text{ km} \times \frac{1}{1000} \text{ km} \times \frac{1}{1000} \text{ km} \
= \frac{2300}{1000 000 000} \text{ km}^3 \
= \frac{23}{10 000 000} \text{ km}^3 \
= 0.000 002 3 \text{ km}^3
\]

There are some special units that are important depending on what course you study. For example:

- Surveyors use the hectare (abbreviated ha) which is 100 m \( \times \) 100 m = 10 000 m\(^2\).
- Nurses use the millilitre (abbreviated mL) which is 0.001 litres or 1 cm\(^3\).
- Mining engineers use the tonne (abbreviated t) which is 1000 kilograms.

**Activity 1.1**

In the following problems, make the indicated conversions.

1. 2.5 m to mm  
2. 40 kg to g  
3. 10 mm to km  
4. 350 g to kg  
5. 6.2 A to mA  
6. 0.005 A to mA  
7. 10 cm\(^2\) to mm\(^2\)  
8. 5 ha to m\(^2\)  
9. 34.8 mm\(^3\) to cm\(^3\)  
10. 5 mL to mm\(^3\)  
11. 2 GL to ML  
12. 36 km/h to m/s

Check your answers at the end of this Section.
1.2 Rounding

When making measurements we use different degrees of accuracy depending on the circumstances.

If some one asks you, ‘When did you arrive?’, you might give an answer like ‘3 pm’ rather than ‘3.09’ pm.

On the other hand, if you want to know when a bus departs from the bus station the timetable gives the time correct to the minute – such as 5.12 pm, which means 12 minutes past five.

Consider the marker on the scale below.

If you were required to give the measurement correct to the nearest integer, the answer would be 33 as the marker is clearly closer to the 33 than the 32 mark in the scale.

But if you wanted a higher degree of accuracy, say to the first decimal place, then you need to examine the location of the marker more closely.

The interval from 32 to 33 is subdivided into ten smaller intervals at 32.1, 32.2, 32.3, 32.4, and so on up to 32.9 and 33.0.

The marker lies between 32.6 and 32.7 but is closer to 32.7, so correct to one decimal place the measurement is 32.7.

We could have known this if we had been given the measurement to an even greater degree of accuracy, say 32.695. The 9 in the second decimal place’s position means the measurement is closer to 32.7 than it is 32.6. The 9 is more than half way.

In general, to round a measurement or a calculated result to a specified place:

• note the digit to the right of that specified place
• if it is more than 5, or 5 followed by non-zero digits, increase the digit in the specified place by 1
• if it is less than 5, leave the digit in the specified place unaltered
• if it is 5 and there are no digits following, or just zeros, leave the digit in the specified place unaltered if it is even but increase it by 1 if it is odd
• eliminate all digits to the right of the specified place
The tables in the following examples illustrate these rules.

**Example 1**

<table>
<thead>
<tr>
<th>Measured or Calculated Value</th>
<th>Correct to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 dp</td>
</tr>
<tr>
<td>1.092 4</td>
<td>1.1</td>
</tr>
<tr>
<td>8.685 9</td>
<td>8.7</td>
</tr>
<tr>
<td>15.593 7</td>
<td>15.6</td>
</tr>
<tr>
<td>87.362 5</td>
<td>87.4</td>
</tr>
<tr>
<td>4.097 2</td>
<td>4.1</td>
</tr>
<tr>
<td>28.248 9</td>
<td>28.2</td>
</tr>
<tr>
<td>12.999 9</td>
<td>13.0</td>
</tr>
</tbody>
</table>

**Example 2**

<table>
<thead>
<tr>
<th>Measured or Calculated Value</th>
<th>Correct to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nearest 10</td>
</tr>
<tr>
<td>12.650 1</td>
<td>10</td>
</tr>
<tr>
<td>75.650 0</td>
<td>80</td>
</tr>
<tr>
<td>15.500</td>
<td>20</td>
</tr>
<tr>
<td>182.500</td>
<td>180</td>
</tr>
</tbody>
</table>

**Activity 1.2**

1. Complete the following table.

<table>
<thead>
<tr>
<th>Measured or Calculated Value</th>
<th>Correct to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 dp</td>
</tr>
<tr>
<td>12.237 4</td>
<td></td>
</tr>
<tr>
<td>8.362 6</td>
<td></td>
</tr>
<tr>
<td>10.527 3</td>
<td></td>
</tr>
<tr>
<td>15.020 5</td>
<td></td>
</tr>
<tr>
<td>4.968 5</td>
<td></td>
</tr>
<tr>
<td>7.359 1</td>
<td></td>
</tr>
<tr>
<td>18.699 8</td>
<td></td>
</tr>
<tr>
<td>5.090 8</td>
<td></td>
</tr>
<tr>
<td>3.760 2</td>
<td></td>
</tr>
</tbody>
</table>
2. Complete the table.

<table>
<thead>
<tr>
<th>Measured or Calculated Value</th>
<th>Correct to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nearest 1 000</td>
</tr>
<tr>
<td>219</td>
<td>0</td>
</tr>
<tr>
<td>5 988</td>
<td></td>
</tr>
<tr>
<td>999</td>
<td></td>
</tr>
<tr>
<td>1 250</td>
<td></td>
</tr>
<tr>
<td>1 350</td>
<td></td>
</tr>
<tr>
<td>550</td>
<td></td>
</tr>
<tr>
<td>35 765</td>
<td></td>
</tr>
<tr>
<td>4 450</td>
<td></td>
</tr>
<tr>
<td>500</td>
<td></td>
</tr>
</tbody>
</table>

Check your answers at the end of this Section.

1.3 Significant figures

Instead of using a given number of decimal places to express the accuracy of a measurement or calculation, significant figures (SF) are sometimes used. This especially happens in science.

Using significant figures is easy unless zeros are involved. This is because in the decimal system, zero is not only used as a digit to indicate the number of units, tens, hundredths etc as in the number 2004 but is also used to place a decimal point such as in 0.02. Note that there is no difference between 0.02 m and 2 cm; they both mean the same.

In addition, zero is used to indicate accuracy such as in the measurement 20.50 m which is measured to two decimal places and is a measurement between 20.495 m and 20.505 m. This is a more accurate measurement than 20.5 m which is only measured to one decimal place and is a measurement between 20.45 m and 20.55 m. Lastly, zeros are used in rounding. If a commentator states that there are 66 000 people at a footy match, he or she has probably rounded the number to the nearest 1000 to produce a nice ‘round’ figure.

The above observations lead to the following rules:

- **All non-zero digits are significant.**

  **Example 1** A number such as 12.274 has five significant figures. It is said to be correct to five significant figures. When corrected back to 12.27 it is correct to four significant figures. 12.3 is correct to three significant figures.
• **Zeros between significant figures are themselves significant.**
  
  **Example 2**  A number such as 309 has three significant figures. Similarly 1.03 has three significant figures.

• **Zeroes used to indicate the position of the decimal point are not significant.**
  
  **Example 3**  0.000 000 462 and 0.0405 both contain three significant figures.

• **Zeros at the end of the decimal part of a decimal number are significant.**
  
  **Example 4**  34.60 has four significant figures while 5.160 has four significant figures and 2.00 has three significant figures.

• **Zeros at the end of a whole number (i.e., a number without a decimal part) are significant unless they are the result of rounding.**
  
  **Example 5**  20 as in ‘there are 20 plants in the garden’ has two significant figures but 12 000 as in ‘there are approximately 12 000 plants in the field’ has two significant figures.

Correcting to a number of significant figures is done using the same rules as correcting to a number of decimal places. This is illustrated in the following example.

**Example 6**  Correct the given number to the indicated number of significant figures.

<table>
<thead>
<tr>
<th>Number</th>
<th>No. of SF</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.1925</td>
<td>4</td>
<td>8.192</td>
</tr>
<tr>
<td>8.1925</td>
<td>2</td>
<td>8.2</td>
</tr>
<tr>
<td>0.008 26</td>
<td>2</td>
<td>0.008 3</td>
</tr>
<tr>
<td>0.008 26</td>
<td>1</td>
<td>0.008</td>
</tr>
<tr>
<td>30 275</td>
<td>4</td>
<td>30 280</td>
</tr>
<tr>
<td>30 275</td>
<td>3</td>
<td>30 300</td>
</tr>
<tr>
<td>1.99</td>
<td>2</td>
<td>2.0</td>
</tr>
<tr>
<td>1.99</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>15.206</td>
<td>2</td>
<td>15</td>
</tr>
</tbody>
</table>
Activity 1.3

1. How many significant figures are there in each of the following?
   a) 39.7  b) 5  c) 4.54
d) 0.003 9  e) 2.2  f) 5.013

2. Give the best three significant figure approximation for each of the following:
   a) 12.89  b) 1.623  c) 78.04  d) 0.020 034

3. Complete the table by rounding the number on the left to the accuracy indicated in the middle column.

<table>
<thead>
<tr>
<th>Number</th>
<th>No. of SF.</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 865 820</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>24 865 820</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>24 865 820</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.008 357 1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>0.008 357 1</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>0.008 357 1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>21.987</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Check your answers at the end of this Section.

1.4 Estimation

In performing calculations using a calculator, incorrect answers are often obtained because data has been entered incorrectly into the calculator. For this reason, it is a good practice to develop the habit of estimating an approximate answer. This should be done using suitable rounding. The exact answer can then be obtained by keying in the numerical data. If the results match closely, it is unlikely that a mistake has been made.

A few examples will make the method clear.

**Note:** The symbol ≈ means ‘is approximately equal to’.

**Example 1** For the multiplication $19 \times 333$

   a) Estimate an answer showing working.
   b) Calculate the answer using a calculator.

   a) $19 \approx 20$ rounded to the nearest 10
      $333 \approx 330$ rounded to the nearest 10
      $20 \times 330 \approx 6 600$ estimated to the nearest 100
   b) $19 \times 333 = 6 327$
Example 2  Estimate an answer for \( \frac{21 \times 95}{11.2} \)

\[
21 = 20 \quad \text{rounding down}
\]

\[
95 = 100 \quad \text{rounding up to compensate}
\]

\[
11.2 \approx 10 \quad \text{rounding down, to produce a ‘large’ estimate}
\]

Thus a suitable estimate would be \( \frac{20 \times 100}{11.2} = 200 \)

The exact answer is 178.125.

Activity 1.4

For each of the following:

a) Estimate an answer and show working.

b) Calculate the answer using a calculator to the appropriate number of significant digits/decimal places.

1. \( 0.347 \div 0.113 \) to three significant digits

2. \( 33297 \times 0.0357 \) to four significant digits

3. \( 6211.9 \div 0.0357 \) to three significant digits

4. \( 0.006392 \times 0.0436 \div 0.000239 \) to four decimal places

5. \( 0.0139 \times 0.25 \div 0.46 \times 0.08 \times 4 \) to three decimal places

Check your answers at the end of this Section.

1.5 The scientific calculator

For this subject, a scientific calculator is sufficient. There are basically two types, the most recent type being the algebraic scientific calculator. An algebraic scientific calculator allows you to type in calculations in the order in which they have been written down. What you have written down generally stays on the screen. On older scientific calculators, you need to press the mathematical operation key after you have entered the number. If you have one of this type, it is time to upgrade because the algebraic type is much more user-friendly.

Many students now have a graphics calculator and are quite familiar with its various capabilities. Be aware, however, that merely passing on information from such a calculator generally will not be sufficient to obtain full marks on a test question. Also, don’t expect your teacher to be an expert on the particular brand of graphics calculator you have bought.
When you buy a new calculator, make sure that you keep your manual in a safe place. If you have lost yours, the Internet can help out because most of the major manufacturers have websites that allow you to download manuals.

There are many types of errors students make on their calculator but the most common are:

- doing a trigonometry question in the wrong angle measurement type (You must make sure that you know how to change between degrees and radians.)
- forgetting to check that your answer is reasonable (You should always try and double-check all calculations in case you have pressed the wrong button.)
- rounding before the end of a calculation (Store calculations in the memory and use all the decimal places during calculations. If you use a rounded value too soon, you will lose accuracy.)
- forgetting to use brackets on division calculations eg when dividing by ALL the bottom part of a fraction
- not including brackets when the square root of a complex expression needs to be taken.

Many calculators are now very powerful and have amazing computational power. Some of the programmable graphics calculators are minicomputers. Although they will all calculate with 100% accuracy every time, unfortunately they are only as good and as accurate as their operator.

**Activity 1.5**

Calculate the following. Give your answer to 2 decimal places if rounding is required.

1. \(5000 \times 0.075 \times 1.75\)
2. \(2.2 \times 60 \times 0.15 \div 61\)
3. \((3.2)^2\)
4. \((0.25)^2\)
5. \((8.3)^3\)
6. \(\sqrt{17.2}\)
7. \(\frac{1}{0.016}\)
8. \(\frac{1}{\sqrt{137.4}}\)
9. \((-0.25)^2\)
10. \((-1.24)^2\)
11. \(4^{2.43} \times 2^{1.23}\)
12. \(42.5^{0.64} \times 25.2^{0.23}\)
13. \(\frac{27.32 \times 2.13}{21.4 - 32.4}\)
14. \(\frac{27.13 \div 5.21^2 - (32.1 - 23.9)^2}{12.3 \times 9.81^{1.25}}\)
15. \( \sqrt{6.432 \times (18.12 + 3.74)^2} \)

16. \( \frac{\sqrt{1.243 + 1.024^2}}{4.24 - 1.05} \)

17. \( \sqrt[4]{32} \)

18. \( 2\sqrt[4]{98.6799876} \)

Check your answers at the end of this Section.

1.6 Types of numbers

In various branches of mathematics different sets of numbers are used.

In fact, all numbers we deal with are part of the set of **Real Numbers**. Important subsets of this set are:

- the set of **Counting Numbers** ie \( \{1, 2, 3, 4, \ldots\} \)
- the set of **Whole Numbers** ie \( \{0, 1, 2, 3, 4, \ldots\} \)
- the set of **Integers** ie \( \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \)
- the set of **Rational Numbers**. These numbers can be expressed as the ratio \( \frac{a}{b} \) where \( a \) and \( b \) are integers, \( b \neq 0 \). Examples are \( \frac{7}{8}, \frac{4}{1}, -0.237 \left( = \frac{-237}{1000} \right) \) and \( 0.6 \left( = \frac{2}{3} \right) \)
- the set of **Irrational Numbers**. These numbers cannot be expressed as the ratio of two integers. Examples are \( \sqrt{2}, \sqrt{56.8} \) and \( \pi \).

How these numbers relate is best illustrated by means of a diagram.

```
<table>
<thead>
<tr>
<th>Real Numbers</th>
<th>Irrational Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rational Numbers</td>
<td>Fractions</td>
</tr>
<tr>
<td>Integers</td>
<td>Negative Integers</td>
</tr>
<tr>
<td>Whole Numbers</td>
<td>Zero</td>
</tr>
<tr>
<td>Counting Numbers</td>
<td></td>
</tr>
</tbody>
</table>
```

Fig 1.1
An example of this classification follows.

**Example**  List  

- a) the integers,  
- b) the rational numbers  

in the set \{-10, 4, 0.5, \sqrt{7}, 0.2, 0, \sqrt{5}\}

- a) \(-10, 4, 0\) and \(\sqrt{5}\)  
  \((\sqrt{5} = 3, \text{ hence is an integer})\)

- b) all except \(\sqrt{7}\)  
  \((\text{All terminating and repeating decimals are rational.})\)

**Activity 1.6**

1. List the positive integers less than 16.
2. A prime number is a counting number that can only be divided by 1 and itself. The number 1 is not considered a prime number. List the set of prime numbers among the first 100 counting numbers.
3. State whether the following are true (T) or false (F):
   - a) The square of a prime number is a prime number
   - b) \(\sqrt{5}\) is an irrational number
   - c) 3 is a counting number
   - d) \(-9\) is a rational number
   - e) \(\frac{4}{5}\) is an integer
   - f) 0.555 555 55 is a decimal fraction

**Check your answers at the end of this Section.**
1.7 Order of operation

When a numerical expression involves a number of different operations, they must be dealt with in a particular order. This order is determined by the following rules.

- Work out the content of any brackets, starting with the innermost bracket.
- Multiply any quantities connected by the word ‘of’ (from left to right).
- Do multiplication and division (from left to right).
- Do addition and subtraction.

An easy way to remember this is the mnemonic

**B I M D A S**

(Brackets, Indices, Multiplication, Division, Addition, Subtraction).

**Example 1** Evaluate $7 + 9 ÷ 3 + 4 \times 6$

\[
7 + 9 ÷ 3 + 4 \times 6 = 7 + (9 ÷ 3) + (4 \times 6)
\]

(after inserting brackets to account for order)

\[
= 7 + 3 + 24
\]

\[
= 34
\]

**Example 2** Evaluate $200 - [190 - (93 + 5)]$

\[
200 - [190 - (93 + 5)] = 200 - [190 - 98]
\]

(inner brackets first)

\[
= 200 - 92
\]

\[
= 108
\]

**Activity 1.7**

Calculate the following:

1. $7 + 3 \times 4 - 9 ÷ 3$
2. $8 \times 4 + 2 \times 6 ÷ 3$
3. $[(5 - 2) + (81 ÷ 3)] - 4$
4. $[400 - 24\% \text{ of } (300 - 200 \times 1.2)] \times 1.2 + 8$

Check your answers at the end of this Section.
1.8 Signed numbers

Signed numbers or directed numbers are positive or negative numbers used to represent quantities that are opposite to each other. For example if +5 represents 5 °C above zero, then −5 would represent 5 °C below zero. Note that while a minus sign (−) must be used to indicate a negative number, a positive number may be shown by a plus sign (+) or no sign at all. Thus 5 °C may be represented by +5 or 5.

The following table shows other pairs of opposites that may be represented by +5 and −5:

<table>
<thead>
<tr>
<th>+5</th>
<th>−5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5\text{ deposited}$</td>
<td>$5\text{ withdrawn}$</td>
</tr>
<tr>
<td>5 km/h faster</td>
<td>5 km/h slower</td>
</tr>
<tr>
<td>5 shots above par</td>
<td>5 shots below par</td>
</tr>
<tr>
<td>5 m north</td>
<td>5 m south</td>
</tr>
<tr>
<td>5 kg gained</td>
<td>5 kg lost</td>
</tr>
<tr>
<td>$5\text{ credit}$</td>
<td>$5\text{ debit}$</td>
</tr>
</tbody>
</table>

How to do arithmetic with signed numbers is illustrated below.

We will let + mean 'saving money' and − mean 'spending money'.

Adding signed numbers

When we add, we combine numbers.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Formal Notation</th>
<th>Actual Notation</th>
<th>Answer</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Last week I saved $5 and this week I saved $10.</td>
<td>(+5) + (+10)</td>
<td>5 + 10</td>
<td>15</td>
<td>I have $15 more than last week.</td>
</tr>
<tr>
<td>Last week I saved $5 and this week I spent $10.</td>
<td>(+5) + (−10)</td>
<td>5 −10</td>
<td>−5</td>
<td>I have $5 less than last week.</td>
</tr>
<tr>
<td>Last week I spent $5 and this week I saved $10.</td>
<td>(−5) + (+10)</td>
<td>−5 + 10</td>
<td>5</td>
<td>I have $5 more than last week.</td>
</tr>
<tr>
<td>Last week I spent $5 and this week I spent $10.</td>
<td>(−5) + (−10)</td>
<td>−5 −10</td>
<td>−15</td>
<td>I have $15 less than last week.</td>
</tr>
</tbody>
</table>

These examples illustrate that to add numbers:

- add the numbers if they have the same sign and put the common sign in front
- subtract the numbers if they do not have the same sign and put the sign of the largest in front.
Subtracting signed numbers

When we subtract, we find the difference between the numbers.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Formal Notation</th>
<th>Actual Notation</th>
<th>Answer</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>The difference between saving $5 and saving $10</td>
<td>(+5) – (+10)</td>
<td>5 – 10</td>
<td>-5</td>
<td>I am $5 less well off.</td>
</tr>
<tr>
<td>The difference between saving $5 and spending $10</td>
<td>(+5) – (-10)</td>
<td>5 – 10</td>
<td>15</td>
<td>I am $15 better off.</td>
</tr>
<tr>
<td>The difference between spending $5 and saving $10</td>
<td>(-5) – (+10)</td>
<td>-5 – 10</td>
<td>-15</td>
<td>I am $15 less well off.</td>
</tr>
<tr>
<td>The difference between spending $5 and spending $10</td>
<td>(-5) – (-10)</td>
<td>-5 – (-10)</td>
<td>5</td>
<td>I am $5 better off.</td>
</tr>
</tbody>
</table>

These examples illustrate that:
• to subtract a signed number, add its opposite.

Multiplying signed numbers

Multiplication is repeated addition.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Formal Notation</th>
<th>Actual Notation</th>
<th>Answer</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>I save $5 per week. 3 weeks from now I will have:</td>
<td>(+3) × (+5)</td>
<td>3 × 5</td>
<td>15</td>
<td>I will have $15 more than now.</td>
</tr>
<tr>
<td>I spend $5 per week. 3 weeks from now I will have:</td>
<td>(+3) × (-5)</td>
<td>3 × (-5)</td>
<td>-15</td>
<td>I will have $15 less than now.</td>
</tr>
<tr>
<td>I save $5 per week. 3 weeks ago I had:</td>
<td>(-3) × (+5)</td>
<td>-3 × 5</td>
<td>-15</td>
<td>I had $15 less than now.</td>
</tr>
<tr>
<td>I spend $5 per week. 3 weeks ago I had:</td>
<td>(-3) × (-5)</td>
<td>-3 × (-5)</td>
<td>15</td>
<td>I had $15 more than now.</td>
</tr>
</tbody>
</table>

These examples illustrate that when two signed numbers are multiplied:
• the product is positive if the numbers have the same sign
• the product is negative if the numbers have different signs.
**Dividing signed numbers**

Division finds how many times a given number is contained in another number.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Formal Notation</th>
<th>Actual Notation</th>
<th>Answer</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>I have saved $15 over 3 weeks. Per week I have saved an average of:</td>
<td>(+15) ÷ (+3)</td>
<td>15 ÷ 3</td>
<td>5</td>
<td>I have saved an average of $5 per week.</td>
</tr>
<tr>
<td>How many weeks ago did I have $15 more than I have now if I have been spending $5 per week?</td>
<td>(+15) ÷ (-5)</td>
<td>+15 ÷ (-5)</td>
<td>-3</td>
<td>3 weeks ago I had $15 more than I have now.</td>
</tr>
<tr>
<td>I have spent $15 over 3 weeks. Per week I have an average of:</td>
<td>(-15) ÷ (+3)</td>
<td>-15 ÷ 3</td>
<td>-5</td>
<td>I have spent an average of $5 per week.</td>
</tr>
<tr>
<td>For how many weeks have I spent on average $5 per week if I have spent a total of $15?</td>
<td>(-15) ÷ (-5)</td>
<td>-15 ÷ (-5)</td>
<td>3</td>
<td>For 3 weeks I have spent an average of $5 per week.</td>
</tr>
</tbody>
</table>

These examples illustrate that when two signed numbers are divided:
- the quotient is positive if the numbers have the same sign
- the quotient is negative if the numbers have different signs.

Let’s practise some examples using these rules.

**Example 1** Add +27, -15, +3 and -5.

Method 1: Adding successively
\[
(+27) + (-15) + (+3) + (-5) = (+12) + (+3) + (-5) = (+15) + (-5) = (+10)
\]

Method 2: Adding by sign
\[
(+27) + (-15) + (+3) + (-5) = (+30) + (-20) = (+10)
\]

**Example 2** Calculate (+8) – (-12) – (+3) + (-5)
\[
(+8) – (-12) – (+3) + (-5) = (+8) + (+12) + (-3) + (-5) = (+20) + (-8) = (+12) or 12
\]
Example 3  Simplify $8 - 9 - 10 + 2$

$$8 - 9 - 10 + 2 = (+8) + (-9) + (-10) + (+2)$$

$$= (+10) + - (19)$$

$$= (-9) \text{ or } -9$$

Example 4  Calculate $(-1) \times (+2) \times (-3) \times (-4)$

$$(-1) \times (+2) \times (-3) \times (-4) = (-2) \times (+12)$$

$$= (-24) \text{ or } -24$$

Example 5  Simplify $\frac{-4 \times (-2) \times 5}{-5 \times 8}$

$$\frac{-4 \times (-2) \times 5}{-5 \times 8} = \frac{8 \times 5}{-40} = -1$$

Activity 1.8

1. Calculate the following accurately to 3 decimal places.
   a) $(-5) + (+7)$
   b) $(-3) - (-9)$
   c) $(-3) \times (-3)$
   d) $(-12) \div (-4) \times (+2)$
   e) $-8 + 3 - 7$
   f) $-2 \times 5 \times (-7) \times (-8)$
   g) $\frac{-2 \times 3}{-6}$
   h) $\frac{-3 \times 7 + (9 \times (-1))}{(7 + 3) \times (-5)}$

2. Add $-9$, $+23$, $-17$ and $+15$

3. Calculate where $n$ is a whole number.
   a) $(-1)^3 \text{ ie } (-1) \times (-1) \times (-1)$
   b) $(-1)^4$
   c) $(-1)^{100}$
   d) $(-1)^{2n+1}$

Check your answers at the end of this Section.
1.9 Scientific and engineering notation

Numbers such as 10, 100, 1000, etc can be expressed as a string of tens multiplied together. Such numbers are called powers of ten and they can be expressed with the power like this:

\[
\begin{align*}
100 &= 10 \times 10 = 10^2 \\
1000 &= 10 \times 10 \times 10 = 10^3 \\
10,000 &= 10 \times 10 \times 10 \times 10 = 10^4
\end{align*}
\]

For numbers such as \(\frac{1}{100}\) and \(\frac{1}{1000}\) this string of tens multiplied together occurs on the bottom line of a fraction. We indicate this in ten power by using a negative sign like this:

For example \(0.001 = \frac{1}{1000} = \frac{1}{10 \times 10 \times 10} = \frac{1}{10^3} = 10^{-3}\)

In some areas of technology, science and engineering, numbers which are very large or very small occur. In order to express these numbers easily we use a power of ten notation.

There are two commonly used notations using powers of ten. They are called scientific (or standard) notation and engineering notation.

Scientific notation

To express a number in scientific notation we use a power of ten such that the number part is between one and ten.

**Example 1** Express 1 500 in scientific notation

\[1 \,500 = 1.5 \times 10^3\]

To do these conversions quickly, put the decimal point where it has to be so that the number is between one and ten and then multiply by a power of ten that would move it back to where it is in the original number.

Remember: the decimal point moves one place to the right each time a ten is multiplied.

**Example 2** Express 35 000 in scientific notation.

\[35 \,000 = 3.500 \times 10^4\]

(We need to move the decimal point 4 places to the right.)

For small numbers, we must recall that dividing by ten moves the decimal point to the left.

**Example 3** Express 0.002 35 in scientific notation.

\[0.002 \,35 = 2.35 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 2.35 \times 10^{-3}\]

(We need to move the decimal point 3 places to the left.)
Engineering notation
To express a number using engineering notation we use a power of ten that is a multiple of three ie $-9, -6, -3, 0, 3, 6, 9$ and so on while the number part must be between 1 and 1 000.

Example 4  Convert (i) 82 000 and (ii) 0.000 25 to engineering notation.

(i)  82 000 = 82 × 1000 = 82 × 10³

(ii) 0.000 25 = $\frac{250}{1 000 000} = 250 \times 10^{-6}$

Engineering notation is very useful in converting between units in the metric system.

Example 5  A city’s water consumption in a month is 6 590 000 litres. Express this in megalitres.

6 590 000 L = 6.59 × 10⁶ litres

= 6.59 megalitres

= 6.59 ML.

Numbers that are already in scientific or engineering notation can be expressed as ordinary decimals by simply using the power of ten to shift the decimal point.

Example 6  Express (i) $2.7 \times 10^4$ and (ii) $125 \times 10^{-3}$ as an ordinary decimal:

(i)  $2.7 \times 10^4 = 2.7 \times 10 \times 10 \times 10 \times 10 = 27 000$  (Decimal point moves 4 places to the right.)

(ii)  $125 \times 10^{-3} = \frac{125}{10 \times 10 \times 10} = 0.125$  (Decimal point moves 3 places to the left.)

All scientific calculators can convert between the different types of notation. In fact, you can enter numbers into your calculator in the different formats. You should refer to your instruction booklet because each calculator operates slightly differently.

Activity 1.9
1. Convert into scientific notation.
   a)  250 000  b)  0.000 000 000 000 000 000 000 987

2. Express 0.000 082 in engineering notation.

3. Express in ordinary decimal form.
   a)  $7.25 \times 10^4$  b)  $436 \times 10^{-6}$

4. Use your calculator to evaluate the following expressions:
   a)  $(2.75 \times 10^4) \div (1.52 \times 10^{-3})$

   b)  $\sqrt{45636} \times \sqrt{0.8765}$
Express the result in scientific notation correct to 3 significant figures.

5. Express $\sqrt{2.4 \times 10^{17}}$ in
   a) scientific notation     b) engineering notation
   Use 2 decimal place accuracy.

   **Check your answers at the end of this Section.**
Assessment 1

1. Convert 0.000 923 kg to mg.
2. Convert 23.4 cm³ to mm³.
3. How many mL are there in 1 223.5 mm³?
4. Round 23.65 to one decimal place.
5. Round 34 652 to the nearest 100.
6. How many significant figures are there in 6.050?
7. Round 0.235 correct to 2 significant figures.
8. Give a suitable estimate of \(\frac{32 \times 99}{59}\)
9. Calculate correct to two decimal places \(\sqrt{\frac{23.56 + 3.412 \times (-3.2)^2}{23.56 - 5.61}}\)
10. Is 0.343 434… a rational number?
11. List 10 non-negative integers.
12. Calculate \((5 + 7 \times 8) - \left(4 - \left[\frac{6}{2 \times 3 - 2}\right]^3\right)\)
13. Add \(-5, +4, -6, -8, +4\) and \(-2\)
14. Express 0.000 000 087 in scientific notation.
15. Calculate \((4.2 \times 10^3) \div (32.9 \times 10^{-6})\) expressing your answer in engineering notation.

Answers to activities

Activity 1.1

1. 2 500 mm
2. 40 000 g
3. 0.00001 km
4. 0.35 kg
5. 6200 mA
6. 5 mA
7. 1 000 mm²
8. 50 000 m²
9. 0.0348 cm³
10. 5 000 mm³
11. 2 000 ML
12. 10 m/s
Activity 1.2

1.

<table>
<thead>
<tr>
<th>Measured or Calculated Value</th>
<th>Corrected to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 dp</td>
</tr>
<tr>
<td>12.237 4</td>
<td>12.2</td>
</tr>
<tr>
<td>8.362 6</td>
<td>8.4</td>
</tr>
<tr>
<td>10.527 3</td>
<td>10.5</td>
</tr>
<tr>
<td>15.020 5</td>
<td>15.0</td>
</tr>
<tr>
<td>4.968 5</td>
<td>5.0</td>
</tr>
<tr>
<td>7.359 1</td>
<td>7.4</td>
</tr>
<tr>
<td>18.699 8</td>
<td>18.7</td>
</tr>
<tr>
<td>5.090 8</td>
<td>5.1</td>
</tr>
<tr>
<td>3.760 2</td>
<td>3.8</td>
</tr>
</tbody>
</table>

2.

<table>
<thead>
<tr>
<th>Measured or Calculated Value</th>
<th>Corrected to:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nearest 1 000</td>
</tr>
<tr>
<td>219</td>
<td>0</td>
</tr>
<tr>
<td>5 988</td>
<td>6 000</td>
</tr>
<tr>
<td>999</td>
<td>1 000</td>
</tr>
<tr>
<td>1 250</td>
<td>1 000</td>
</tr>
<tr>
<td>1 350</td>
<td>1 000</td>
</tr>
<tr>
<td>550</td>
<td>1 000</td>
</tr>
<tr>
<td>35 765</td>
<td>36 000</td>
</tr>
<tr>
<td>4 450</td>
<td>4 000</td>
</tr>
<tr>
<td>500</td>
<td>0</td>
</tr>
</tbody>
</table>

Activity 1.3

1. a) 3  b) 1  
   c) 3  d) 2  
   e) 2  f) 4

2. a) 12.9  b) 1.62  
   c) 78.0  d) 0.0200
Apply mathematical techniques in a manufacturing, engineering or related environment

MEM30012A

3.

<table>
<thead>
<tr>
<th>Number</th>
<th>No. of S.F.</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>24 865 820</td>
<td>6</td>
<td>24 865 800</td>
</tr>
<tr>
<td>24 865 820</td>
<td>4</td>
<td>24 870 000</td>
</tr>
<tr>
<td>24 865 820</td>
<td>2</td>
<td>25 000 000</td>
</tr>
<tr>
<td>0.008 357 1</td>
<td>4</td>
<td>0.008 357</td>
</tr>
<tr>
<td>0.008 357 1</td>
<td>3</td>
<td>0.008 36</td>
</tr>
<tr>
<td>0.008 357 1</td>
<td>2</td>
<td>0.008 4</td>
</tr>
<tr>
<td>21.987</td>
<td>2</td>
<td>22</td>
</tr>
</tbody>
</table>

Activity 1.4

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 3</td>
<td>3.07</td>
</tr>
<tr>
<td>2. 1 200</td>
<td>1 189</td>
</tr>
<tr>
<td>3. 150 000</td>
<td>174 000</td>
</tr>
<tr>
<td>4. 1 600</td>
<td>1460.5803</td>
</tr>
<tr>
<td>5. 0.15</td>
<td>0.024</td>
</tr>
</tbody>
</table>

Activity 1.5

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 656.25</td>
<td>2. 0.32</td>
</tr>
<tr>
<td>3. 10.24</td>
<td>4. 16</td>
</tr>
<tr>
<td>5. 14.09</td>
<td>6. 4.15</td>
</tr>
<tr>
<td>7. 62.5</td>
<td>8. 0.09</td>
</tr>
<tr>
<td>9. 0.06</td>
<td>10. 0.65</td>
</tr>
<tr>
<td>11. 68.12</td>
<td>12. 0.19</td>
</tr>
<tr>
<td>13. –5.29</td>
<td>14. –0.38</td>
</tr>
<tr>
<td>15. 13.60</td>
<td>16. 1.31</td>
</tr>
<tr>
<td>17. 2</td>
<td>18. 1.73</td>
</tr>
</tbody>
</table>
Activity 1.6
1. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
3. a) F b) T
c) T d) T
e) F f) T

Activity 1.7
1. 16 2. 36
3. 26 4. 470.72

Activity 1.8
1. a) 2 b) 6
c) 9 d) 6
e) –12 f) –560
g) 1 h) –0.047 (3 d.p)
2. 12
3. a) –1 b) 1
c) 1 d) –1

Activity 1.9
1. a) $2.5 \times 10^5$ b) $9.87 \times 10^{-22}$
2. $82 \times 10^{-4}$
3. a) 72 500 b) 0.000 436
4. a) $1.81 \times 10^{-1}$ b) $2.00 \times 10^2$
5. a) $2.21 \times 10^4$ b) $22.13 \times 10^3$
Section 2 – Algebra

2.1 Substitution

In algebra, letters are used to represent numbers. By using letters and mathematical symbols, lengthy verbal statements can be represented by short algebraic statements called literal expressions.

Consider the following examples:

<table>
<thead>
<tr>
<th>Verbal Statement</th>
<th>Literal Statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Six added to a number</td>
<td>$6 + a$</td>
</tr>
<tr>
<td>The sum of a number and 4</td>
<td>$x + 4$</td>
</tr>
<tr>
<td>7 less a number</td>
<td>$7 - x$</td>
</tr>
<tr>
<td>The difference of two numbers</td>
<td>$p - q$</td>
</tr>
<tr>
<td>The quotient of a number and 3</td>
<td>$\frac{q}{3}$</td>
</tr>
<tr>
<td>The product of 8 and a number</td>
<td>$8 \times a$</td>
</tr>
<tr>
<td>The square of a number</td>
<td>$t \times t$</td>
</tr>
</tbody>
</table>

Note that any letter can be used to represent the unknown number but usually lower case letters are used.

Also note that the multiplication sign is usually omitted. Thus $8 \times a$ is normally written as $8a$. Similarly $3x^2y$ means $3 \times x^2 \times y$ or $3 \times x \times x \times y$.

If we assign a particular value to a letter, generally called variable, (or better, pronumeral), a literal expression can be evaluated.

Example 1 Evaluate $2x - 4$ for $x = 6$

$$2x - 4 = 2(6) - 4$$

$$= 12 - 4$$

$$= 8$$

Note that in the above example we have used the order of operation BIMDAS rule which we encountered before.

If a particular literal expression contains more than one variable such as $x$ and $y$, you should systematically replace each letter with its given numerical value.

Remember that when you multiply like signs you get a positive whereas when you multiply unlike signs you get a negative.
Example 2  Evaluate $2x^2(x - y) - 3xy$ when $x = -4$ and $y = 2$.

\[
2x^2(x - y) - 3xy = 2(-4)^2(-4 - 2) - 3(-4)(2)
\]

\[
= 2(16)(-6) - 3(-8)
\]

\[
= -192 - (-24)
\]

\[
= -168
\]

Activity 2.1
Evaluate each of the following for $x = 2$, $y = -3$ and $z = 0$.

1. $2x - y$
2. $3x^2 + 2y$
3. $\frac{x^2y}{3x}$
4. $3xyz$
5. $\frac{1}{2}(x - 2y)$

Check your answers at the end of this Section.

2.2  Simplification of algebraic expressions

A literal expression such as $3x$ consists of only one term and is called a monomial. On the other hand $8 + a$ has two terms; it is a binomial.

Although $3x$ is only a monomial, it has two parts. The 3 is called the numerical coefficient while $x$ is referred to as the literal coefficient.

Like terms are terms that have the same literal coefficient. Thus $2x^2z$ and $-3x^2z$ are like terms. $5x^2z$ and $2xz^2$, however, are not like since the first monomial has literal coefficient $x^2 \times z$ while the literal coefficient of the second monomial is $x \times z^2$.

Sometimes literal expressions may be simplified. However, the golden rule of algebra should always be observed. This rule states that:

Only like terms can be added or subtracted.

This rule makes sense if you consider a situation in which you receive $2$ and then a further $3$. By using the letter $d$ to stand for a dollar coin, you could write that you have received $2d + 3d = 5d$.

Had you received $2$ and then a further 50 cents and letting $c$ stand for cents, you would have received $2d + 50c$ and there is no way of simplifying this further unless you change 50c to $\frac{1}{2}d$. It certainly would not become $52cd$ or any other such nonsense combination.

The following examples further illustrate that, when adding or subtracting like terms, you should add or subtract the numerical coefficients but keep the same literal coefficient.
Example 1  Simplify $2d + 3d + 10c$

$$2d + 3d + 10c = 5d + 10c$$

Example 2  Simplify $3x - 5xy - 2x + 7xy + 4$

$$3x - 5xy - 2x + 7xy + 4 = 3x - 2x - 5xy + 7xy + 4$$

$$= 1x + 2xy + 4$$

$$= x + 2xy + 4$$

It can happen that an expression does not contain any like terms. In that case, we say that the expression 'cannot be simplified'.

Activity 2.2

Simplify the following.

1. $5x + 3y - y + 2x + 2y - x$
2. $5a + 3b - 2a + b - 3a + 4b$
3. $3x^2 - 2x + 4x - 2x^2 + 3x - 5x$
4. $5x^2 - 2xy - xy - x^2$
5. $5a - 4ax - x^2 - 4x - 3a + x^2 - 2a + 4ax$
6. $-3x^2 - 2x + 3 - 4x + 2x^2 - 3x - 2$
7. $2a - 3b + 4c - 7$

Check your answers at the end of this Section.

2.3  Removal of brackets

If we have to find the sum of $(x + 3)$ and $(x + 3)$ we could proceed in two different ways.

1. Writing out in full produces \(x + 3 + x + 3 = x + x + 3 + 3 = 2x + 6\)
2. Adding like terms produces $2(x + 3)$

So clearly $2(x + 3) = 2x + 6$.

In algebraic language, we have distributed 2 over $(x + 3)$. This distributive law can be summarised as:

$$a(b + c) = ab + ac$$

Thus when applying the distributive law, the expression outside the bracket must be multiplied in turn with each expression inside the brackets.
Example 1  Write \(-(b - c)\) without brackets.
\[-(b - c) = (-1)(b - c) = (-1)(b) - (-1)(c)\]
\[= -b - (-c)\]
\[= -b + c\]

You observe from the above example that a negative in front of the brackets, changes the sign of each term inside the brackets.

Example 2  Simplify \(2(x - y) - (x - y)\)
\[2(x - y) - (x - y) = 2x - 2y - x + y\]
\[= 2x - x - 2y + y\]
\[= x - y\]

Example 3  Simplify \((x + 5)(x + 4)\)
\[(x + 5)(x + 4) = (x + 5)(x) = (x + 5)(4)\]
\[= x^2 + 5x + 4x + 20\]
\[= x^2 + 9x + 20\]

Activity 2.3
Write each of the following without brackets by applying the distributive rule.
1. \(3(x - 5)\)
2. \(-4(2y - 5)\)
3. \(-(2y - 1)\)
4. \(-8(x - 2)\)
5. \(-3(x + 2y - 1)\)
6. \(-2(x - y) - (x + y)\)
7. \(2x(3x^2 - 2x + 1)\)
8. \((x + 1)(x + 4)\)

Check your answers at the end of this Section.

2.4 Linear equations
An equation is a statement with an equality sign (\(=\)). A linear equation is of the form \(ax + b = 0\). This means that:
- There must be an \(=\) sign.
- There is one pronumeral, sometimes called the unknown or, wrongly, variable.
- The highest power of the pronumeral is 1.

Example 1  Which of the following are linear equations?

a) \(2x + 7\)

b) \(2x + 7 = 0\)

c) \(2x + 7 = 6\)

d) \(x = 7\)

e) \(x^2 - 3x + 7 = 0\)

f) \(x + y = 0\)
a) No  No = sign
b) Yes  a = 2 and b = 7
c) Yes  Can be changed to $2x + 1 = 0$  ie a = 2 and b = 1
   (See later topic.)
d) Yes  Can be changed to $x - 7 = 0$  ie a = 1 and b = -7
   (See later topic.)
e) No  Power of pronumeral is not 1 but 2 (a square)
f) No  Two variables

To solve an equation is to find the (truth) value of the unknown that makes the equation true. For example if we solve $2x - 8 = 0$, we will obtain $x = 4$.

Essentially there are two rules for solving linear equations. The rules are Cross Multiplication rule and the Transposing rule.

We shall consider each in turn and then combine them in more difficult problems.

### The cross multiplication rule

Consider the following examples:

**Example 2**  Let $x$ be the cost of an ice cream.

If 5 ice creams cost $4.50 calculate $x$.

Here $5x = $4.50  or  $x = \frac{4.50}{5} = $0.90

**Example 3**  Let $x$ be the cost of an ice cream. If 5 ice creams cost $C$, calculate $x$.

Here $5x = C$  or  $x = \frac{C}{5}$

**Example 4**  Let $x$ be the cost of an ice cream. If $n$ ice creams cost $C$, calculate $x$.

Here $nx = C$  or  $x = \frac{C}{n}$

The above examples illustrate the ‘cross multiplication rule’:

- If $ax = b$ then $x = \frac{b}{a}$
- If $\frac{x}{a} = b$ then $x = ab$

The rule is called the cross multiplication rule because we multiply crosswise over the = sign.
Example 5  Solve the equation $5x = 10$

If $5x = 10$ then $x = \frac{10}{5}$

Hence $x = \frac{18}{-1} = -18$

Cross multiply 5

Example 6  Solve the equation $-\frac{x}{3} = 6$

If $-\frac{x}{3} = 6$ then $-x = 6 \times 3 = 18$  Cross multiply 3

Hence $x = \frac{18}{-1} = -18$  Cross multiply $-1$

Note: You should always check by substitution that you have obtained the correct solution.

Thus in example 5, substituting $x = 2$ gives $5 \times 2 = 10$ which is obviously correct.

In example 6, substituting $-18$ gives $\frac{-18}{3} = -(-6) = 6$ which is also correct.

The transposing rule
Consider the following examples:

Example 7  Let $x$ be the cost of an ice cream. If the cost of an ice cream and $5$ equals $5.90$ calculate $x$.

Here $x + 5 = 5.90$  or  $x = 5.90 - 5 = 0.90$

Example 8  Let $\$x$ be the cost of an ice cream. If the cost of an ice cream and $\$C$ equals $5.90$ calculate $x$.

Here $x + \$C = 5.90$  or  $x = 5.90 - \$C$
Example 9  Let $$/x$$ be the cost of an ice cream. If the cost of an ice cream and $$$/C$$ equals $$$/P$$, calculate $$x$$.

Here $$x + $$$/C$$ = $$$/P$$ or $$x = $$$/P$$ - $$$/C$$

This last example, without the $$$$ sign, reads that when $$x + C = P$$ then $$x = P - C$$. Notice that the $$C$$ has shifted to the other side of the equation 9 (is transposed) but has changed from positive to negative ie has changed sign. This is the transposing rule.

- If $$x + a = b$$ then $$x = b - a$$
- If $$x - a = b$$ then $$x = b + a$$

Thus when you change sides, you must change signs.

Example 10  Solve the equation  $$x - 3 = 7$$

If $$x - 3 = 7$$ then $$x = 7 + 3$$ Transpose $$-3$$.

Hence $$x = 10$$

Example 11  Solve the equation  $$6 - x = 11$$

If $$6 - x = 11$$ then $$6 - 11 = x$$ Transpose both $$x$$ and $$11$$.

or $$-5 = x$$

Hence $$x = -5$$ Transpose both $$x$$ and $$5$$.

Another way of solving this last problem would have been to transpose the $$6$$ to obtain $$-x = 11 - 6$$ or $$-x = 5$$ which is equivalent to $$(-1)x = 5$$. To obtain $$x$$ we could then have cross multiplied $$(-1)$$ to obtain $$x = \frac{5}{-1} = -5$$

As before, always check your answers by substitution.

Activity 2.4

Solve the following equations.

1. $$2x = 6$$
2. $$\frac{x}{2} = 6$$
3. $$x + 2 = 6$$
4. $$x - 2 = 6$$
5. $$-3x = 12$$
6. $$\frac{-x}{3} = 8$$
7. $$-x - 12 = 20$$
8. $$3 + (-x) = 8$$
9. $$\frac{2x}{5} = 8$$
10. $$2x - 8 = 18$$

Check your answers at the end of this Section.
2.5 More difficult linear equations

To solve the last problem of the Activity 2.4, you would have had to use both the cross multiplication and the transposing method. Hopefully, you would have transposed first. In fact, this is one of the rules for solving more difficult linear equation problems. Let us state these rules followed by an example for each rule.

 transpose to bring like terms together.

**Example 1**

Solve \(2x - 1 = x + 5\)

If \(2x - 1 = x + 5\) then \(2x - x = 5 + 1\) Transpose \(x\) and \(-1\).

Hence \(x = 6\) Check by substitution!

In general, transpose first before cross multiplying.

**Example 2**

Solve \(2x + 3 = 7\)

If \(2x + 3 = 7\) then \(2x = 7 - 3\) Transpose 3.

or \(2x = 4\)

or \(x = \frac{4}{2}\) Cross multiply 2.

Hence \(x = 2\) Check by substitution!

In general, it is best to remove brackets first.

**Example 3**

Solve \(2(x + 3) = 7\)

If \(2(x + 3) = 7\) then \(2x + 6 = 7\)

or \(2x = 7 - 6 = 1\)

Hence \(x = \frac{1}{2}\)

Remove fractions by multiplying both sides of the equation by the lowest common denominator (LCD).

**Example 4**

Solve \(\frac{x}{2} + 1 = \frac{x}{3}\)

If \(\frac{x}{2} + 1 = \frac{x}{3}\) then \(6\left(\frac{x}{2} + 1\right) = 6\left(\frac{x}{3}\right)\) The LCD of 2 and 3 is 6.

or \(6\left(\frac{x}{2}\right) + 6(1) = 6\left(\frac{x}{3}\right)\) Multiply each term by 6.

or \(3x + 6 = 2x\)

Hence \(x = -6\)
Activity 2.5
Solve the following equations.

1. \(5x + 12 = 4x + 2\)
2. \(-y - 2 = -3y + 12\)
3. \(3x + 4 = 20 - x\)
4. \(2(x + 3) = -2\)
5. \(-3(2m - 4) = 6\)
6. \(\frac{x - 2}{3} = 8\)
7. \(6(2x + 1) = 3(x + 8)\)
8. \(\frac{x}{2} + 1 = -2\)
9. \(4(x - 2) - 6(x - 4) = 26\)
10. \(\frac{2x - 1}{2} + 3 = \frac{-2}{3}(x + 2)\)

Check your answers at the end of this Section.

2.6 Literal expressions

In word problems, the mathematics is ‘hidden’ in words and the words need to be ‘translated’ into mathematics.

The secret to translating word problems is to know the meaning of ‘key’ words that indicate certain mathematical operations. Here is a partial list.

<table>
<thead>
<tr>
<th>Mathematical Operator</th>
<th>Special Words</th>
<th>Mathematical Operator</th>
<th>Special Words</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition</td>
<td>sum</td>
<td>Multiplication</td>
<td>product of</td>
</tr>
<tr>
<td></td>
<td>increased by</td>
<td>of</td>
<td>times, multiplied by</td>
</tr>
<tr>
<td></td>
<td>more than</td>
<td>of</td>
<td>increased/decreased by</td>
</tr>
<tr>
<td></td>
<td>combined, together</td>
<td></td>
<td>by a factor</td>
</tr>
<tr>
<td></td>
<td>total</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>added to</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>difference between/of</td>
<td>Division</td>
<td>quotient</td>
</tr>
<tr>
<td>Subtraction</td>
<td>decreased by</td>
<td></td>
<td>per, at</td>
</tr>
<tr>
<td></td>
<td>less than, fewer than</td>
<td></td>
<td>out of</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>ratio of</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>percent (divide by 100)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>average</td>
</tr>
</tbody>
</table>
**Example 1**  Translate the expressions on the left. Let \( x \) be the number.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sum of a number and 8</td>
<td>( x + 8 )</td>
</tr>
<tr>
<td>The difference between a number and 8</td>
<td>( 8 - x )</td>
</tr>
<tr>
<td>8 less than a number</td>
<td>( x - 8 )</td>
</tr>
<tr>
<td>The ratio of a number and 8</td>
<td>( \frac{x}{8} )</td>
</tr>
<tr>
<td>8 less then the total of a number and 2</td>
<td>( (x + 2) - 8 )</td>
</tr>
<tr>
<td>The sum of three consecutive whole numbers</td>
<td>( x + (x + 1) + (x + 2) )</td>
</tr>
<tr>
<td>The average profit per item if 8 items are sold</td>
<td>( \frac{x}{8} )</td>
</tr>
</tbody>
</table>

**Example 2**  Twenty litres of petrol were poured into two containers of different size. Express the amount poured into the smaller container in terms of the amount poured in the larger one.

Let \( p \) be the amount poured in the larger container. Then the remainder \( 20 - p \) is poured into the smaller container.

**Example 3**  A rectangle has one side 5 cm longer than the other. Find an expression for the area.

Let \( x \) be the size of the smaller side. Then \( x + 5 \) is the size of the larger side. Hence the area is \( x(x + 5) \) cm\(^2\) or \( x^2 + 5x \) cm\(^2\).

**Example 4**  At a football match, adults pay $22 and children $12. If three times as many adults attended as children, express the total taking in terms of the number of children who attend.

Let \( n \) be the number of children who attend. Then \( 3n \) is the number of adults who attend.

Each child pays $12. Hence the takings for children is: \( n \times 12 = 12n \)

Each adult pays $22. Hence the takings for adults is: \( 3n \times 22 = 66n \)

Therefore the total takings is \( 12n + 66n = 78n \)
Activity 2.6
Find an expression for the following statements.
1. The distance covered by a car in 3 hours travelling at $x$ km/h.
2. Bob’s age $n$ years from now if he is now $m$ years of age.
3. The number of minutes in $t$ hours.
4. When a number is increased by 15, its value is doubled.
5. The cost of 3 bananas and 7 oranges if a banana cost $b$ cents and an orange cost 10 cents less than a banana.
6. The ‘future value’ of an amount of money earning 5% per annum simple interest, invested over 4 years.

Hint: To increase by 5% is to multiply by 1.05.

Check your answers at the end of this Section.

2.7 Linear equation word problems
To solve word problems, follow these steps.

• Work in an organised manner. Draw a diagram or picture if appropriate.
• Clearly define the unknown and choose a logical letter to represent it.
• Using ‘key’ words, write an equation that expresses the relationship between the numbers in the problem.
• Solve the equation and check that the solution makes sense.
• Answer the problem.

Example 1  Syd is 5 years older than Ernie. Their ages add to 27. How old is Syd?
Let Syd be $s$ years of age. Then Ernie is $s - 5$.

Now $s + (s - 5) = 27$ or $2s - 5 = 27$

or $2s = 32$ giving $s = 16$

Thus Syd is 16 years of age.

Example 2  Bill had originally $54 and then saved $12 per month until he had $150. For how many months did he save?

Let $m$ be the number of months he saved.

Then $54 + 12m = 150$ or $12m = 150 - 54 = 96$ or $m = \frac{96}{12} = 8$

Hence Bill saved for 8 months.
Example 3  A rectangle is twice as long as it is wide. If the perimeter is 48 cm, find the area.

Let $w$ be width. Then the length is $2w$.

Now $2(2w + w) = 48$ or $2(3w) = 48$.

Then $6w = 48$, giving $w = \frac{48}{6} = 8$

Thus the rectangle has dimensions 8 cm by 16 cm.

Hence its area is $8 \text{ cm} \times 16 \text{ cm} = 128 \text{ cm}^2$.

We finish with a very difficult example from chemistry.

Example 4  How much pure acid should be added to 24 L of a 35% solution (a solution with 35% pure acid) to obtain a 50% solution?

Let $x$ be the number of litres of pure acid to be added. Then the final volume will be $(24 + x)$ litres of 50% acid. Now:

Pure acid in 24 L of 35% solution + Pure acid = Pure acid in final 50% solution.

This gives $0.35 \times 24 + x = 0.5 \times (24 + x)$

or $8.4 + x = 12 + 0.5x$

or $x - 0.5x = 12 - 8.4$

or $0.5x = 3.6$

Thus $x = \frac{3.6}{0.5} = 7.2$

Hence, 7.2 L of pure acid must be added.

Activity 2.7

Solve the following problems.

1. A woman is 30 years older than her daughter. Together they are 88 years of age. How old is the daughter?

2. A room is 2 m longer than it is wide. If the perimeter of the room is 16 m, find the area.

3. The sum of three consecutive whole numbers is 60. Find the middle number.

4. Subtract 5 from a number and double the result. If the final number is three quarters of the original number, what is the original number?

5. Bananas cost 10c more than oranges. Three bananas and three oranges cost a total of $3.30. Find the cost of one orange.

6. How many litres of a 20% solution of alcohol should be added to 40 L of an 80% solution to obtain a solution that contains 60% alcohol?

Check your answers at the end of this Section.
2.8 Simultaneous linear equations

In the last sections we solved linear equations with one variable and found one unique solution. An equation involving two variables such as \( x + y = 10 \) has many different solutions. Examples are \( x = 5 \) with \( y = 5 \) or \( x = 2 \) with \( y = 8 \). However, if another equation connecting the two variables is given, there is generally one specific solution.

For example, if in addition to \( x + y = 10 \) it is also known that \( x - y = 2 \) the solution could only be \( x = 6 \) with \( y = 4 \). We have solved a pair of simultaneous equations.

There are many methods available for solving a pair of simultaneous equations but we shall use one of the easier methods called the elimination method.

This method depends on writing both equations in the form with the variables on the left and the number on the right as was done in the example above. We then add or subtract the equations to eliminate either the terms involving \( x \) or the terms involving \( y \). The resulting equation will then have only one variable and can be solved using the method described earlier.

An example will make this clear.

**Example 1**  Solve the pair of equations \( x + 5y = 6 \) and \( x + 2y = 3 \)

Arrange the equations as follows:

\[
\begin{align*}
\text{Original Equations:} & \\
\text{A} : & \quad x + 5y = 6 \\
\text{B} : & \quad x + 2y = 3
\end{align*}
\]

Subtract:

\[
\begin{align*}
\text{A} - \text{B} : & \\
(5y - 2y) & = (6 - 3) \\
3y & = 3
\end{align*}
\]

This gives \( y = 1 \)

Now substitute this value for \( y \) into either equation.

Using \( x + 5y = 6 \) gives \( x + 5(1) = 6 \)

or \( x + 5 = 6 \)

So \( x = 1 \).

Hence the solution is \( x = 1 \) and \( y = 1 \).

It is generally worthwhile to check the solution by substituting the values into the second equation. In the above example \( 1 + 2(1) = 3 \) which is correct.

Note that the solution is frequently written as an ordered pair which is of the form \((x, y)\). Thus in the above example we could have written \((1, 1)\).
Consider another example.

**Example 2**  Solve $3x + 4y = 6$ simultaneously with $5x - 4y = 2$

Add:

\[
\begin{align*}
3x + 4y &= 6 \\
5x - 4y &= 2 \\
8x &= 8 \\
\end{align*}
\]

Now $8x = 8$ gives $x = 1$.

Substituting into the first equation gives

$3(1) + 4y = 6$

or $4y = 3$

So $y = \frac{3}{4}$

Hence the solution is $x = 1$ and $y = \frac{3}{4}$ or $(1, \frac{3}{4})$.

In certain problems, we cannot eliminate either the $x$ or $y$ terms by adding or subtracting the equation unless we multiply one or both equations by a suitable constant.

Consider this example:

**Example 3**  Solve the pair of simultaneous equations $7x + 4y = -2$

$5x + 2y = 2$

Multiplying the second equation by 2 produces the equation $10x + 4y = 4$.

We now proceed as before ie

\[
\begin{align*}
7x + 4y &= -2 \\
10x + 4y &= 4 \\
-3x &= -6 \\
\end{align*}
\]

This gives $x = 2$.

On substitution we obtain

$7(2) + 4y = -2$ or $14 + 4y = -2$

or $4y = -16$

So $y = -4$

Hence the solution is $x = 2$ and $y = -4$ or $(2, -4)$.
Activity 2.8
Solve the following equations for \( x \) and \( y \) and check the solutions:

1. \( x + y = 7 \)
   \( x - y = 1 \)
2. \( x - 3y = 7 \)
   \( x + y = 3 \)
3. \( 3x + 2y = 7 \)
   \( x + y = 3 \)
4. \( 4x - 3y = 1 \)
   \( x + 3y = 19 \)
5. \( 7x - 4y = 37 \)
   \( 6x + 3y = 51 \)
6. \( 4x - 6y = -2.5 \)
   \( 7x - 5y = -0.25 \)

Check your answers at the end of this Section.

2.9 Simultaneous equations word problems

As with solving word problems involving just one variable, you should define the variables clearly using logical letters such as \( d \) for distance. Then write down the equations that connect the variables and solve the set of simultaneous equations. Finally, clearly state your answer, using correct units if applicable.

Study the following examples.

Example 1  
Jim is 15 years older than Bill and their combined ages is 75 years. How old is each?

Let Bill’s age be \( B \) and Jim’s age be \( J \). Then \( J - B = 15 \) and \( J + B = 75 \)

Solving these equations in the usual way we get

\[
\begin{align*}
J - B &= 15 \\
J + B &= 75 \\
2J &= 90
\end{align*}
\]

It follows that \( J = 45 \). Then \( 45 - B = 15 \) or \( B = 30 \)

Hence Jim is 45 years of age and Bill is 30 years of age.

Example 2  
The perimeter of a rectangle is 22 m while the difference between the length and the width is 3 m. Find the dimensions.

Let \( \ell \) be the length and \( w \) be the width.

Now \( 2\ell + 2w = 22 \) and \( \ell - w = 3 \)

or \( 2\ell + 2w = 22 \)

\[
\begin{align*}
2\ell - 2w &= 6 \\
4\ell &= 28
\end{align*}
\]

This gives \( \ell = 7 \) from which it follows that \( w = 4 \).

Hence the dimensions are 4 m by 7 m.

We finish with a very difficult problem, especially in terms of setting up the equations.
Example 3  Max has two bags of toys, one for boys and the other for girls. If he gave each boy three toys and each girl two, he can supply 44 children, while if he gave each boy two and each girl one he can supply 78 children. How many toys are there in each bag?

Let $x$ be the number of toys in the boys’ bag
$y$ be the number of toys in the girls’ bag.

Now $\frac{x}{3} + \frac{y}{2} = 44 \quad \text{(1)}$ and $\frac{x}{2} + y = 78 \quad \text{(2)}$

Multiplying equation (1) by 6 and equation (2) by 2 produces
$2x + 3y = 264$ and $x + 2y = 156$ which lead to the solution
$x = 60$ and $y = 48$.

Thus there are 60 presents in the bag for boys and 48 in the bag for girls.

Activity 2.9

1. A foreman and seven men together earn $1040 per day whilst two foremen and 17 men together earn $2464 per day. Find the earnings for a foreman and for a man.

2. For one installation, 8 ceiling roses and 6 plugs are required. The total cost of these items is $160. For a second installation, 12 ceiling roses and 5 plugs are used and the cost is $176. Find the cost of a ceiling rose and a plug.

3. Two quantities $x$ and $y$ are connected by the law $y = ax + b$. When $x = 2$, $y = 13$ and when $x = 5$, $y = 22$. Find the values of $a$ and $b$.

4. In a certain lifting machine it is found that the effort, $E$, and the load, $W$ are connected by the equation $E = aW + b$. An effort of 2.6 raises a load of 8, whilst an effort of 3.8 raises a load of 12. Find the values of the constants $a$ and $b$ and determine the effort required to raise a load of 15.

5. If 100 m of wire and 8 plugs cost $62 and 150 m of wire and 10 plugs cost $90 find the cost of 1 m of wire and the cost of a plug.

6. Find two numbers such that their sum is 27 and their difference is 3.

7. The ‘Two Tell’ telephone company charges a fixed cost for rental and a fixed charge per local call. You receive two bills, one for $65 for 200 calls and the other for $70 for 240 calls. What is the rental and what is the local call charge?

8. A father is 24 years older than his son. In three years’ time he will be three times as old as his son. How old is each now?

Check your answers at the end of this Section.
2.10 Formulae evaluation

Consider the rule for finding the area of a rectangle.

\[ \text{Area of Rectangle} = \text{length units} \times \text{breadth units} \]

This rule may be abbreviated by letting:

- Area be represented by \( A \)
- Length be represented by \( L \) and
- Breadth be represented by \( B \)

Then the rule becomes: \( A = L \times B \) or, using our new knowledge of algebra, \( A = LB \).

In this rule, more commonly called formula, the letter \( A \) is called the subject of the formula. The subject is always the letter that is on its own on the left hand side of the formula.

A subject of a formula may be evaluated by substituting the values of the other symbols that make up the formula.

Consider the following examples in which ‘real’ formulae are used where all symbols have actual units attached. If you studied, for example, physics, you would be expected to supply the correct units with your answer but that is not necessarily required here.

**Example 1** Calculate the circumference (\( C \)) of a circle, correct to 1 decimal place, using the formula \( C = 2\pi r \), given that \( \pi = 3.14 \) and \( r = 6 \text{ cm} \)

By the rules of algebra \( C = 2\pi r \) means \( C = 2 \times \pi \times r \)

Thus \( C = 2 \times 3.14 \times 6 = 37.68 \)

Hence \( C = 37.7 \text{ cm} \)

**Example 2** Given \( E = \frac{1}{2}mv^2 \), find \( E \) given \( m = 4.5 \) and \( v = 4.3 \)

Substituting we obtain \( E = \frac{1}{2} \times 4.5 \times 4.3^2 \)

\[ = 41.6025 \]

\[ = 41.6 \]

(A reasonable answer would be 41.6 so that we do not introduce more accuracy than is given by the actual measurements \( m \) and \( v \).)
Example 3  The time in seconds of a complete swing of a simple pendulum is given by the formula \( T = 2\pi\sqrt{\frac{\ell}{g}} \).

Find, correct to 2 decimal places, the time of a complete swing of length \( \ell = 0.13 \text{ m} \) using \( \pi = 3.142 \) and \( g = 9.81 \text{ m/s}^2 \).

Substituting:  
\[
T = 2 \times 3.142 \times \sqrt{\frac{0.13}{9.81}} \\
= 6.284 \times 0.11511... \\
= 0.723391 \\
= 0.72 \text{ (2 decimal places)}
\]

Thus it takes 0.72 seconds for a complete swing.

Activity 2.10

1. Given \( V = \frac{1}{2}BHS \), find \( V \) when \( B = 4 \), \( H = 2.5 \) and \( S = 7 \)

2. Find \( A \) using the formula \( A = 2\pi(r + h) \) with \( \pi = 3.14 \), \( r = 6 \) and \( h = 13.5 \)

3. The power \( P \) in an electrical circuit may be expressed by the formula \( P = \frac{V^2}{R} \)

   Evaluate \( P \) correct to 2 decimal places given that \( V = 24.62 \) and \( R = 45.21 \)

4. Find \( r \), using 2 significant digits, given that \( r = \frac{A^2 + H^2}{2H} \), \( A = 3.7 \) and \( H = 2.5 \)

5. If \( y = ax^2 + bx + c \), the two values of \( x \) (if they exist) for which \( y \) becomes zero can be found from the 'quadratic formula'

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Consider \( 3x^2 + 5x - 7 = 0 \).

Calculate the two values of \( x \) (one using + and the other using −) correct to two decimal places.

Check your answers at the end of this Section.
2.11 Transposing a formula

We sometimes have to evaluate a symbol which is not the subject of a formula.

For example, we may need to evaluate the value of \( m \) in the formula \( F = ma \) when we know the values of \( F \) and \( a \).

One method of obtaining the value of \( m \) is to re-arrange the formula and to make \( m \) the subject. This process is called **transposing the formula**. Sometimes we say that we need to **solve the formula** for \( m \).

The steps in transposing a formula are identical to solving an equation.

Compare the steps involved in the following table.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Formula</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 18 = 3 + 5x )</td>
<td>Solve ( v = u + at ) for ( t )</td>
<td>Swapping sides</td>
</tr>
<tr>
<td>( 5x + 3 = 18 )</td>
<td>( u + at = v )</td>
<td>Isolating unknown</td>
</tr>
<tr>
<td>( 5x = 18 - 3 )</td>
<td>( at = v - u )</td>
<td>( v - u ) must be left since the values are unknown</td>
</tr>
<tr>
<td>( 5x = 15 )</td>
<td>( at = v - u )</td>
<td></td>
</tr>
<tr>
<td>( x = \frac{15}{5} )</td>
<td>( t = \frac{v - u}{a} )</td>
<td>Again no simplification is possible because the values of ( v, u ) and ( t ) are unknown.</td>
</tr>
<tr>
<td>( x = 3 )</td>
<td>( t = \frac{v - u}{a} )</td>
<td></td>
</tr>
</tbody>
</table>

As in the above illustration, two or more different operations may need to be used. The order in which this is done is important.

Consider this example:

**Example 1** Make \( C \) the subject of the formula \( F = \frac{9}{5}C + 32 \)

\[
\begin{align*}
\text{In this example, it may be tempting to cross multiply the 5 but this should only be done after the term which involves the new subject is isolated.} \\
\text{Thus first write } F - 32 = \frac{9}{5}C \text{ or } \frac{9}{5}C = F - 32 \\
(\text{Swapping sides}) \\
\text{Now cross multiplication produces } C = \frac{5}{9}(F - 32)
\end{align*}
\]

Whenever a subject is contained within brackets it is advisable to isolate this set of brackets on one side of the equal sign and then proceed as before.
This is illustrated in the following example.

**Example 2**  Make \( r \) the subject in the formula \( A = S(1 - r) \)

Cross multiplying \( S \) gives \( \frac{A}{S} = 1 - r \) (Brackets are no longer required.)

Then transposing terms gives the final result \( r = 1 - \frac{A}{S} \)

**Note:** Another method involves multiplying out \( S \) first to give \( A = S - Sr \)

Transposing terms would then produce \( Sr = S - A \)

Then cross multiplying \( S \) gives \( r = \frac{S - A}{S} \)

This, at first glance, looks different from our first answer but is not if it is realised that a fraction may be ‘split up’ ie

\[
\frac{S - A}{S} = \frac{S}{S} - \frac{A}{S} \quad \text{or} \quad r = 1 - \frac{A}{S}
\]

**Activity 2.11**

1. In \( I = \frac{PRT}{100} \) make \( R \) the subject.
2. Solve the formula \( C = A + 2B \) for \( B \).
3. Make \( f \) the subject of the formula
   \[ v = u + \frac{ft}{m} \]
4. Transpose the formula to make \( v \) the subject.
   \[ s = \frac{1}{2}(u + v)t \]
5. Make \( x \) the subject of the formula
   \[ w = \frac{aq}{(2x - p)} \]

Check your answers at the end of this Section.
2.12 More difficult transposition problems

Some formulae contain powers (squares, cubes etc) or surds (square roots, cube roots etc). In other formulae, the symbol that has to be made the subject occurs more than once.

Before we study some examples, please make sure that you do not make one of the following errors that are commonly made:

1. If \( x^2 = a \), it does not mean that the value of \( x \) is just \( \sqrt{a} \). It could also be \( -\sqrt{a} \).
   
   For example if \( x^2 = 25 \), \( x \) could be \( +5 \) or \( -5 \) because \((+5)^2 = (-5)^2 = 25\).

2. \( \sqrt{a} + b \) is not equal to \( \sqrt{a} + \sqrt{b} \).
   
   For example \( \sqrt{36} + \sqrt{64} = \sqrt{100} = 10 \). However \( \sqrt{36} + \sqrt{64} \neq 6 + 8 = 14 \).

3. \( a^2 + b^2 \) is not equal to \( a + b \).
   
   For example \( \sqrt{6^2 + 8^2} = \sqrt{100} = 10 \). However \( 6 + 8 = 14 \).

**Example 1**

Make \( b \) the subject in the formula \( y = b^2 - 3a \).

If \( y = b^2 - 3a \) then \( y + 3a = b^2 \) or \( b^2 = y + 3a \)

Hence we obtain \( b = \sqrt{y + 3a} \) or \( b = -\sqrt{y + 3a} \)

(Sometimes written as \( b = \pm \sqrt{y + 3a} \))

**Example 2**

Make \( s \) the subject in the formula \( v = \sqrt{u^2 + 2as} \)

After squaring both sides we obtain \( v^2 = u^2 + 2as \)

or \( v^2 - u^2 = 2as \)

Then \( 2as = v^2 - u^2 \) (Swapping sides)

or \( s = \frac{v^2 - u^2}{2a} \) (Cross multiplying 2a)

**Example 3**

The molar humidity \( H \) at saturation is given by the formula \( H = \frac{V}{P - V} \)

where \( H \) is ‘total pressure’ and \( V \) is ‘vapour pressure’.

Transpose the formula to make \( V \) the subject.

If \( H = \frac{V}{P - V} \) then \( H(P - V) = V \) or \( HP - HV = V \)

Collecting all terms involving the ‘new’ subject (ie \( V \)) on one side produces \( HP = V + HV \) or \( V + HV = HP \) after swapping sides.

Inserting brackets leaves us with \( V(1 + H) = HP \)

Finally, after cross multiplying \( (1 + H) \) we obtain the final answer

\[ V = \frac{HP}{1 + H} \]
Activity 2.12
1. Let \( v^2 = u^2 + 2as \). Make \( u \) the subject.
2. Solve the formula \( V = \sqrt{2gh} \) for \( h \).
3. Make \( L \) the subject of the formula \( c = (p + q)\sqrt{\frac{L}{g}} \).
4. Make \( r \) the subject of the formula \( m = rp - rt \).
5. Transpose the formula \( V = \frac{Er}{R + r} \) to make
   a) \( E \) the subject
   b) \( r \) the subject.

Check your answers at the end of this Section.

2.13 Indices
We have already dealt with terms such as \( x^2 \) and \( a^3 \). We are now going to look at indices in a more formal way.

We know that \( x^2 = x \times x \) and \( a^3 = a \times a \times a \)

In a similar way we can define \( a^n = a \times a \times a \times a \times a \times a \) \( \ldots \times a \) \((n \text{ lots of } a)\)

In this definition, \( a \) is called the base and \( n \) is an index or exponent.

So-called index rules can easily be established.

- \( a^m \times a^n = a^{m+n} \)
- \( \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \)
- \( \frac{a^m}{a^n} = a^{m-n} \)
- \( a^0 = 1 \)
- \( (a^m)^n = a^{mn} \)
- \( a^{-n} = \frac{1}{a^n} \)
- \( (a \times b)^n = a^n \times b^n \)
- \( \sqrt[n]{a} = a^{\frac{1}{n}} \)

Example 1 Simplify \( a^3 \times a^2 \) (i) by expanding and (ii) using an index rule

(i) Expanding: \( a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a \times a \times a \times a \times a = a^5 \)

(ii) Using: \( a^m \times a^n = a^{m+n} \)

\( a^3 \times a^2 = a^{3+2} = a^{3\cdot2} = a^5 \)
Example 2  Simplify \( a^{-4} + a^{-2} \) (i) by expanding and (ii) using an index rule

(i) Expanding:

\[
\begin{align*}
\frac{a^{-4}}{a^{-2}} &= \frac{1}{a^2} \\
&= a^{-2} \\
&= \left( \frac{1}{a^2} \right)
\end{align*}
\]

(ii) Using: \( a^m \div a^n = a^{m-n} \)

\[
\begin{align*}
a^{-4} \div a^{-2} &= a^{-4-(-2)} \\
&= a^{-2} \\
&= \left( \frac{1}{a^2} \right)
\end{align*}
\]

In problems involving indices, powers of 2, 3 and 5 are frequently used. If you are not familiar with those powers the following table could be handy.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 2^x )</th>
<th>( 3^x )</th>
<th>( 5^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>9</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>27</td>
<td>125</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>81</td>
<td>625</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>243</td>
<td>3 125</td>
</tr>
</tbody>
</table>

Example 3  Express as a single positive power of 2

a) \((2^2)^2\)

b) \(64^2\)

c) \(\sqrt[3]{256}\)

d) \(32^{\frac{5}{3}}\)

a) \((2^2)^2 = 2^4 = \frac{1}{2^4}\)

b) \(64^2 = (2^6)^2 = 2^{12}\)

c) \(\sqrt[3]{256} = (2^8)^{\frac{1}{3}} = 2^4\)

d) \(32^{\frac{5}{3}} = (2^5)^{\frac{5}{3}} = 2^{\frac{5}{3}}\)

Exponential equations are equations in which the unknown occurs as an index. Generally, they are solved by equating base numbers on both sides of the equation and then solving the equation that results by 'cancelling' the base numbers.

An example best illustrates this process.

Example 4  Solve \(5^{x+2} = 125^x\)

\[
5^{x+2} = 125^x \quad \text{or} \quad 5^{x+2} = (5^3)^x
\]

or \(5^{x+2} = 5^{3x}\)

Hence \(x + 2 = 3x\) or \(x - 3x = -2\)

or \(-2x = -2\)

thus \(x = 1\)

As with all problems involving equations, you should check this result by substituting the obtained answer in the original equation.

Here, substituting \(x = 1\) gives \(5^{1+2} = 125^1\) which is true.
Activity 2.13

1. Express
   a) \(n \ldots n\) as a power of \(n\)  
   b) 4096 as a power of 4

2. Simplify
   a) \(d^8 \times d^6\)  
   b) \(e^{2x} \times e^{-3x}\)
   c) \(\frac{3^9}{3^6}\)  
   d) \(\frac{2^{4x-1}}{2^{3-x}}\)

3. Express without brackets
   a) \((mn)^4\)  
   b) \((2x)^3\)
   c) \(\left(\frac{3}{4}\right)^4\)  
   d) \(\left(\frac{r}{5t}\right)^n\)

4. Write as a power of 3
   a) \((3^2)^6\)  
   b) \(81^{x/2}\)
   c) \(\sqrt{3}\)  
   d) \(27^{\frac{1}{3}}\)

5. Find the value of
   a) \(\frac{1}{2^{-1} + 4^{-1}}\)  
   b) \(2^{0.5} \times 32^{0.5}\)

6. By equating base numbers solve
   a) \(2^x = 64\)  
   b) \(3^{x+2} = 81\)
   c) \(2^{-x} = 16\)  
   d) \(5^{2x} = 125^{-2x + 1}\)

Check your answers at the end of this Section.
Assessment 2

1. Simplify \( x - 2x + 3x \)
2. Simplify \( 3x \times 4x \times -2xy \)
3. Remove brackets and simplify: \( 2(x - 2) + 3(x + y) \)
4. Evaluate \( 3x^2y - 5xy - y \) using \( x = 3 \) and \( y = -5 \)
5. Evaluate \( 2(x - y) + 2x^2 - 3(x + y) \) using \( x = -2 \) and \( y = -3 \)
6. Solve the equation \( 2x - 7 = 19 \)
7. Solve \( 3(x - 2) = 4(x - 1) + 8 \)
8. By cross multiplication solve \( \frac{2x - 1}{x + 2} = \frac{3}{4} \)

9. Solve the following word problem:
   A rectangle has sides in the ratio 1 : 2. The perimeter is 102 cm long. How long is the smallest side?
10. Solve the following set of simultaneous equations:
    \( x + 2y = 10 \) and \( x - y = 1 \)
11. By setting up equations solve the following word problem:
    I buy 30 stamps, some for 45 cents each and the rest for 75 cents each. The total I spent is $21. How many of each did I buy?
12. Solve for \( V \) if \( V = 3(AL + B) \) and \( L = 2, A = 5 \) and \( B = -1 \)
13. Calculate \( a \) if \( a = \frac{L_2 - L_1}{L_1(t_2 - t_1)} \) and \( L_1 = 24.1, L_2 = 15.7, t_1 = 9.1 \) and \( t_2 = 6.3 \)
14. Transpose the formula for \( t \) if \( s = \frac{t}{2}(u + v) \)
15. Transpose the formula for \( x \) if \( y = \sqrt[1]{\sqrt{c}} \)
16. Make \( W \) the subject of the formula \( PL = \frac{WR}{1 + W} \)
17. Simplify \( \frac{2^2 \times 2^3}{2^6} \)
18. Solve the indicial equation \( 2^{x + 2} = 4^x \)
Answers to activities

Activity 2.1
1. 7
2. 6
3. −2
4. 0
5. 4

Activity 2.2
1. $6x + 4y$
2. $8b$
3. $x^2$
4. $4x^2 − 3xy$
5. $−4x$
6. $−x^2 − 9x + 1$
7. Cannot be simplified

Activity 2.3
1. $3x − 15$
2. $−8y + 20$
3. $−2y + 1$
4. $−8x + 16$
5. $−3x − 6y + 3$
6. $−3x + y$
7. $6x^2 − 4x^2 + 2x$
8. $x^2 + 5x + 4$

Activity 2.4
1. $x = 3$
2. $x = 12$
3. $x = 4$
4. $x = 8$
5. $x = −4$
6. $x = −24$
7. $x = −32$
8. $x = −5$
9. $x = 20$
10. $x = 13$

Activity 2.5
1. $x = −10$
2. $y = 7$
3. $x = 4$
4. $x = −4$
5. $m = 1$
6. $x = 26$
7. $x = 2$
8. $x = −6$
9. $x = −5$
10. $x = −2.3$

Activity 2.6
1. $3x$
2. $m + n$
3. $60t$
4. $x + 15 = 2x$
5. $3b + 7(b − 10) = 10b − 70$
6. $1.2P$
Activity 2.7
1. 29 years of age
3. 20
5. 50c

Activity 2.8
1. $x = 4$ and $y = 3$
3. $(1, 2)$
5. $x = 7$ and $y = 3$

Activity 2.9
1. Foreman $144$, man $128$
3. $a = 3$ and $b = 7$
5. 1 m wire $0.50$, plug $1.50$
7. Rental $40$, call charge 12.5c

Activity 2.10
1. $V = 35$
3. $P = 13.41$
5. $x = -2.57$ or $x = 0.91$

Activity 2.11
1. $R = \frac{100l}{PT}$
3. $f = \frac{mv^2 - u^2}{t}$
5. $x = \frac{aq + pw}{2w}$

Activity 2.12
1. $u = \pm \sqrt{v^2 - 2as}$
3. $L = g \left( \frac{c}{p + q} \right)^2$
5. (a) $E = \frac{V(R + r)}{r}$
(b) $r = \frac{VR}{E - V}$
Activity 2.13

1. a) \( n^9 \)  
   b) \( 4^6 \)

2. a) \( D^{14} \)  
   b) \( e^{-x} \)  
   c) \( 3^3 = 27 \)  
   d) \( 2^{5x - 4} \)

3. a) \( m^n n^4 \)  
   b) \( 8x^3 \)  
   c) \( \frac{81}{256} \)  
   d) \( \frac{r^n}{s^nt^n} \)

4. a) \( 3^{12} \)  
   b) \( 3^{4x + 8} \)  
   c) \( 3^{0.5} \)  
   d) \( 3 \)

5. a) \( \frac{4}{3} \)  
   b) \( 2^3 = 8 \)

6 a) \( x = 6 \)  
   b) \( x = 2 \)  
   c) \( x = -4 \)  
   d) \( x = 0.375 \)
Section 3 – Geometry

3.1 Names of lines and angles

A point has no size; it is void of quantity and therefore cannot be drawn. Nevertheless, a point is often represented by a small dot. Capital letters are used to name points. Some points have special names. An example is node, the point where two branches of a curve cross in a particular way.

A moving point describes a line. A line has only length, but no breadth. Some special lines are the straight line (usually just called line), ray and line segment.

The table below compares lines, rays and line segments.

<table>
<thead>
<tr>
<th>Line</th>
<th>Ray</th>
<th>Line segment</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Line" /></td>
<td><img src="image" alt="Ray" /></td>
<td><img src="image" alt="Line segment" /></td>
</tr>
<tr>
<td>contains infinitely many points</td>
<td>contains infinitely many points</td>
<td>contains infinitely many points</td>
</tr>
<tr>
<td>no end points</td>
<td>one end point</td>
<td>two end points</td>
</tr>
<tr>
<td>cannot be measured</td>
<td>cannot be measured</td>
<td>can be measured</td>
</tr>
</tbody>
</table>

We can measure the length of a line segment. If a line segment $\overline{AB}$ has length 2 cm, for example, we write $\overline{AB} = 2$ cm.

Points are collinear if they are points of the same line.

An angle is a pair of rays with a common end point called the vertex.
The sign for angle is $\angle$. So in the diagram, $\angle BAC$ is the angle formed when rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ meet in vertex A.

Two angles with a ray in common are called adjacent angles. In the diagram $\angle BAC$ and $\angle CAD$ are adjacent.

### Activity 3.1
1. a) Name this line in 6 different ways.

   ![Diagram](image)

   b) Measure BD in centimetres.

   c) How many line segments can you find? Name them.

2. Name $\angle BAC$ in a different way.

   ![Diagram](image)

3. Draw two intersecting lines $\overrightarrow{AB}$ and $\overrightarrow{CD}$ meeting in X. Correctly name 2 pairs of adjacent angles.

   Check your answers at the end of this Section.

### 3.2 Measuring angles in degrees

The amount of turning between two rays is measured in degrees.

An angle can measure between $0^\circ$ and $360^\circ$. The hands of the clock form an angle at each point in time.
In one hour, the large hand completes one full turn or revolution which has taken it through 360º. This angle is called a *perigon*.

In half an hour, the large hand has completed half a revolution which has taken it through 180º. This is called a *straight angle*.

In a quarter of an hour, the large hand has completed a quarter of a revolution which has taken it through 90º. This is called a *right angle*.

The protractor is a simple instrument that is used to measure angles in degrees. It is divided into 180 equal parts each of which measures 1º.

**Example 1** Using a protractor measure the angle below, then state the:

(i) name of angle
(ii) size of angle (to the nearest degree).

(i) ∠AOB
(ii) 145º

A degree can be divided into 60 minutes. Thus \( \frac{1}{60} \) of 1 degree is 1 minute; written as 1′.

Further, a minute can be divided into 60 seconds. Thus \( \frac{1}{60} \) of 1 minute is 1 second; written as 1″.

As an example of this notation, an angle may have a measurement of 56º 15′ 34″.

Doing arithmetic with angle measurements in degrees, minutes and seconds is rather awkward because the base of angle measurement is 60º; it is a *sexagesimal* system rather than a decimal one.
Example 2  Calculate $51^\circ 40' 23" + 71^\circ 41' 53"$

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Minutes</th>
<th>Seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>40</td>
<td>23</td>
</tr>
<tr>
<td>71</td>
<td>41</td>
<td>53</td>
</tr>
</tbody>
</table>

122

(Carry 1)

122 + 1

(Carry 1)

123

Thus the final answer is $123^\circ 22' 16"$

Note: Most scientific calculators can do this type of angle arithmetic. You should check your manual but the button to use generally looks like the picture on the right.

Also, it is quite common to use decimal degrees, for example $34.876\,0^\circ$. You should be able to change between the two types.

Example 3  Change $123^\circ 22' 16"$ to decimal degrees, correct to 4 decimal places.

$$123^\circ 22' 16" = \left(123 + \frac{22}{60} + \frac{16}{60 \times 60}\right)^\circ$$

$$= 123.3711111...^\circ$$

$$= 123.3711^\circ \text{ to 4 d.p.}$$

Example 4  Change $46.476\,4^\circ$ to degrees, minutes and seconds, to the nearest second.

$$46.476\,4^\circ = 46^\circ + 0.476\,4 \times 60'$$

$$= 46^\circ + 28.584'$$

$$= 46^\circ + 28' + 0.584 \times 60''$$

$$= 46^\circ + 28' + 35.04''$$

$$= 46^\circ\,28'\,35'' \text{ to the nearest second.}$$
Sometimes the sum of two angles either add up to 90°, in which case they are called **complementary angles**, or they add up to 180° in which case they are called **supplementary angles**.

In the diagram

\[\angle J + \angle M = 90^\circ\] \hspace{1cm} \angle J, \angle M are complementary

\[\angle J + \angle P = 180^\circ\] \hspace{1cm} \angle J, \angle P are supplementary

**Example 5** Find the complement of 35° 36’

Subtract 35° 36’ from 90°:

Remember 1° = 60’

\[
\begin{align*}
&90^\circ \quad (90^\circ \text{ becomes } 89^\circ) \\
&- 35^\circ 36’ \quad (\text{Put } 60’ \text{ above } 36’) \\
&89^\circ 60’ \\
&\quad \text{(Take } 1^\circ \text{ from } 90^\circ) \\
&54^\circ 24’ \\
\end{align*}
\]

Thus the complement is 90° – 35° 36’ is 54° 24’

**Example 6** Find the supplement of 51° 40’ 23”

Subtract 51° 40’ 23” from 180°.

Remember 1° = 60’; 1’ = 60”

\[
\begin{align*}
&179^\circ 59’ 60” \quad (\text{Take } 1^\circ \text{ from } 180^\circ) \\
&180^\circ 60” \quad (\text{Take } 1’ \text{ from } 60”) \\
&- 51^\circ 40’ 23” \quad (\text{Then subtract}) \\
&128^\circ 19’ 37” \\
\end{align*}
\]

Thus the supplement of 51° 40’ 23” is 128° 19’ 37”
Activity 3.2

1. Measure each angle with your protractor to the nearest degree.

a) \[ \textcolor{red}{\angle AOB} = \] 

b) \[ \textcolor{red}{\angle AOD} = \] 

c) \[ \textcolor{red}{\angle MOB} = \] 

d) \[ \textcolor{red}{\angle BCA} = \] 

e) \[ \textcolor{red}{\angle AOC} = \]
2. Give the complement of the following angles:
   a) $70^\circ$
   b) $43^\circ$
   c) $10^\circ 30'$
   d) $52^\circ 23'$
   e) $80^\circ 12' 30''$
   f) $36^\circ 26' 24''$

3. Give the supplement of the following angles:
   a) $110^\circ$
   b) $33^\circ$
   c) $135^\circ 30'$
   d) $83^\circ 47'$
   e) $100^\circ 50' 42''$
   f) $10^\circ 19' 33''$

4. Find the:
   a) supplement of $8^\circ 12'$
   b) complement of $0^\circ$
   c) supplement of $43^\circ 52'$
   d) supplement of $113^\circ 13' 4''$
   e) complement of $83^\circ 5'$
   f) supplement of $93^\circ 30' 42''$

5. Simplify the following:
   a) $42^\circ 53' + 46^\circ 18'$
   b) $213^\circ 19' - 83^\circ 45'$
   c) $23^\circ 18' 56'' + 42^\circ 39' 8''$
   d) $360^\circ - 39^\circ 14' 57''$

Check your answers at the end of this Section.

3.3 Radian measure

There is another way of measuring angles which is used extensively in higher mathematics. This method has as its unit the radian which is the standard angular measure in the International System of Units (S.I.).

By definition, 1 radian is the angle between two radii of a circle which cut off (subtend) on the circumference an arc equal in length to one radius.

Thus in the diagram $\theta$ has a measure of 1 radian usually abbreviated as 1 rad or $1^\text{R}$

Radians must be used in certain formulas in mathematics and physics to avoid errors or the need for additional scale factors.

Because of this widespread use, the rad abbreviation or the $^\text{R}$ sign is generally omitted.

To avoid confusion, it has been agreed that the degree sign $^\circ$ must always be written.

Thus when an angle in a formula, such as in the formula for arc length ($s = r\theta$) the angle works out to be 2, it means that the angle is 2 radians in size.
From the diagram you can see that 1 radian must be approximately 60°. In fact, it is approximately 57.3°. The exact conversion is easily derived if you consider that an angle of 360° must have size $2\pi$ rad since the length of the circumference equals $2\pi$ radius distances.

Thus  

$$2\pi \text{ rad} = 360° \quad \text{or} \quad \pi \text{ rad} = 180°$$

or  

$$1 \text{ rad} = \frac{180°}{\pi} \quad \text{after dividing both sides by} \quad \pi$$

Considering that $\pi \approx 3.1415926...$, we obtain  

$$1 \text{ rad} \approx 57.295779...\°$$

$$= 57°17'45'' \text{ to the nearest second}$$

The relationship $\pi \text{ rad} = 180°$ holds the key to converting between the measures because with it the conversion factors can easily be derived. They are summarised in the table.

<table>
<thead>
<tr>
<th>Converting from</th>
<th>Radians to Degrees</th>
<th>Degrees to Radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\times \frac{180°}{\pi^R}$</td>
<td>$\times \frac{\pi^R}{180°}$</td>
<td></td>
</tr>
</tbody>
</table>

Example 1  
Change 4.35 radians to degrees, minutes and seconds.

$$4.35^R = 4.35^R \times \frac{180°}{\pi^R} = 249.23664...°$$

$$= 249°14'12" \text{ (nearest second)}$$

Example 2  
Convert 36° 17' 35'' to radians correct to four decimal places.

$$36°17'35'' = 36.2930555...°$$

$$= 36.2930555...° \times \frac{\pi^R}{180°} = 0.6334^R \text{ (correct to 4 d.p.)}$$

Example 3  
The formula $s = r\theta$ is used to measure arc length(s). Calculate the perimeter of the sector drawn below using 3 significant figures.

![Diagram](image)

$$33° = 33° \times \frac{\pi^R}{180°} = 0.57595...^R$$

Then  

$$s = 6 \times 0.57595... \text{ mm} = 3.46 \text{ mm (3 S.F.)}$$

Thus the perimeter is 6 + 6 + 3.46 = 15.46 mm.
Activity 3.3

1. Convert to radian measure, correct to 4 decimal places.
   a) \( 34^\circ \)  
   b) \( 145.81^\circ \)  
   c) \( 12^\circ 14' \)  
   d) \( 23^\circ 15' 25'' \)  
   e) \( 145^\circ 12' 54'' \)  
   f) \( 56^\circ 15' 34.7'' \)

2. Convert to degrees, correct to the nearest second:
   a) \( 2^R \)  
   b) \( 4.1^R \)  
   c) \( 2.876 5^R \)  
   d) \( \frac{1}{2} \pi^R \)  
   e) \( \frac{\pi^R}{6} \)  
   f) \( \frac{2\pi^R}{3} \)

3. An arc of length 6 cm subtends an angle of 56° at the centre of a circle. Find the length of the radius using the formula \( s = r\theta \).

4. To calculate the area of a sector the formula \( A_{\text{sector}} = \frac{1}{2} r^2 \theta \) where \( \theta \) is the sector angle (or subtended angle) in radians, can be used. Calculate, correct to two decimal places, the area of a sector with radius 5 cm in which the subtended angle is 60°.

Check your answers at the end of this Section.

3.4 Types of angles and angle pairs

We have already defined the perigon (360°), the straight angle (180°) and the right angle (90°). Angles that have a size in between these values are given special names as illustrated in the following table:

<table>
<thead>
<tr>
<th>Name</th>
<th>Range</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acute</td>
<td>less than 90°</td>
<td>25°</td>
</tr>
<tr>
<td>Obtuse</td>
<td>more than 90° but less than 180°</td>
<td>120°</td>
</tr>
<tr>
<td>Reflex</td>
<td>more than 180° but less than 360°</td>
<td>330°</td>
</tr>
</tbody>
</table>

Note: Angles that are not right are collectively called oblique.
When two lines intersect, a special pair of angles is created called **vertically opposite angles**.

In the diagram $\angle AOB$ and $\angle COD$ are vertically opposite and so are $\angle AOC$ and $\angle BOD$.

Vertically opposite angles are equal in all respects or, geometrically speaking, are **congruent**. The symbol that is used is $\cong$.

This is easy to prove using the above diagram. In this diagram $\angle AOB$ and $\angle BOD$ are supplementary but so are $\angle COD$ and $\angle BOD$. Hence $\angle AOB \cong \angle COD$.

**Example 1** Find the value of $a$, $b$ and $c$.

Here $a = 150^\circ$ (supplementary to $30^\circ$)

$b = 30^\circ$ (vertically opposite to $30^\circ$)

$c = 150^\circ$ (vertically opposite to $a$).

When two parallel lines are intersected by a third, called the **transversal**, three different angle pairs are formed. Their names and properties are illustrated in the table on the next page.
Apply mathematical techniques in a manufacturing, engineering or related environment

<table>
<thead>
<tr>
<th>Name</th>
<th>Illustration</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternate</td>
<td></td>
<td>are congruent</td>
</tr>
<tr>
<td>Corresponding</td>
<td></td>
<td>are congruent</td>
</tr>
<tr>
<td>Co-interior</td>
<td></td>
<td>are supplementary</td>
</tr>
</tbody>
</table>

**Example 2**  What is the value of \( x \)?

![Diagram](image)

The angle is in corresponding position to the angle of size 35°.

Hence \( x = 35° \)

(The degree sign must be included with the answer.)
Example 3  What are the values of \(a\), \(b\) and \(c\)?

\[
\begin{align*}
a^\circ & = 53^\circ \quad \text{(corresponding angles)} \\
b^\circ & = 42^\circ \quad \text{(alternate angles)} \\
c^\circ & = 180^\circ - (42^\circ + 53^\circ) \quad \text{(a straight angle)} \\
& = 180^\circ - 95^\circ \\
& = 85^\circ \\
\end{align*}
\]

Hence \(a = 53\), \(b = 42\) and \(c = 85\)

(Note that the \(^\circ\) should not be written because it was already present in the diagram.)

In the above example we saw that \(a + b + c = 180^\circ\) which illustrates that the angle sum in a triangle is \(180^\circ\).

Example 4  Find the values of the pronumerals \(x\) and \(y\).

An additional parallel line must be drawn.

Angle \(y\) is supplementary to \(120^\circ\). Hence \(y = 60\)

The angles which measure \(x^\circ\) and \(y^\circ\) are alternate to the two angles that make up the angle which measures \(110^\circ\). Hence \(x = 50\)
**Activity 3.4**

1. 

a) Name the angle vertically opposite to $\angle AOC$

b) Name the angle(s) adjacent to $\angle COD$

2. Find the sizes of the angles marked with a variable:

   a) 
   
   \[
   x^\circ = 6^\circ \quad 43^\circ 
   \]

   b) 
   
   \[
   40^\circ \quad x^\circ \quad 57^\circ 
   \]

   c) 
   
   \[
   a^\circ \quad b^\circ \quad 15^\circ 
   \]

   d) 
   
   \[
   63^\circ \quad x^\circ 
   \]

   e) 
   
   \[
   x^\circ \quad 35^\circ 
   \]

   f) 
   
   \[
   110^\circ 
   \]

   g) 
   
   \[
   100^\circ \quad \theta 
   \]

   h) 
   
   \[
   a^\circ \quad b^\circ \quad c^\circ \quad 70^\circ \quad 40^\circ 
   \]
3.5 The perimeter of a quadrilateral

The perimeter of a figure is the distance around the figure.

For a quadrilateral, a closed figure made up of four sides, the perimeter is obtained by adding the lengths of all sides.

Example 1 Calculate the perimeter of the quadrilateral in the diagram.

Perimeter \( P \) = 6 m + 8 m + 4 m + 10 m = 28 m

So the length of the perimeter of this quadrilateral is 28 metres.
It is usual to express the perimeter of a polygon in terms of a formula, which is a sort of recipe for the calculation that is to be made.

For a rectangle:

\[ Perimeter = 2 \times (\text{length}) + 2 \times (\text{width}) \]

If we use initial letters instead of names this becomes:

\[ P_{\text{rectangle}} = 2L + 2W \]

Where:
- \( P \) stands for the ‘length of the perimeter’
- \( L \) stands for the ‘length of the rectangle’
- \( W \) stands for the ‘width of the rectangle’

**Example 2** Calculate the length of the perimeter of a rectangle which measures 1200 mm by 1800 mm.

\[
P = 2L + 2W = 2 \times 1200 \text{ mm} + 2 \times 1800 \text{ mm} \\
= 2400 \text{ mm} + 3600 \text{ mm} \\
= 6000 \text{ mm}
\]

So the perimeter is 6000 mm in length.

Here is an applied example where we have to calculate the cost based on the perimeter.

**Example 3** A rectangular table 1850 mm by 950 mm is to have a border trim fixed. The border trim costs $15 per metre, how much will the border cost for the whole table?

\[
P = 2L + 2W = 2 \times 1850 \text{ mm} + 2 \times 950 \text{ mm} \\
= 3700 \text{ mm} + 1900 \text{ mm} \\
= 5600 \text{ mm} = 5.6 \text{ metres}
\]

Now each metre costs $15 so the whole cost will be \( 5.6 \times 15 = 84 \)

**Activity 3.5**

1. Find the perimeter of a rectangle with a length of 4.5 mm and a breadth of 3.5 mm.
2. Find the perimeter of a rectangle with a length of 1 cm and a height of 5 cm.
3. Find the perimeter of a rectangle with a length of 1200 cm and a width of 40 cm.
4. Find the perimeter of a square with sides of 2.7 mm.
5. A table measuring 82 cm by 124 cm has a thin jarrah beading around its edge. What length of beading is used for this table?

Check your answers at the end of this Section.
3.6 The perimeter of a triangle

The perimeter of a triangle is obtained by adding the lengths of the three sides.

**Example 1** Calculate the perimeter of the triangle shown:

\[
\text{Perimeter} = 12.1 \text{ m} + 14.8 \text{ m} + 10.5 \text{ m} = 37.4 \text{ m}
\]

Hence, the perimeter is 37.4 metres.

An equilateral triangle has three sides each of the same length and an isosceles triangle has two sides of equal length. For these special types of triangles we don’t have to know all three side lengths.

**Example 2** Find the perimeter of an equilateral triangle with a side of length 1100 mm.

\[
\text{Perimeter} = 3 \times 1100 \text{ mm} = 3300 \text{ mm}
\]

**Example 3** Calculate the perimeter of this isosceles triangle.

\[
\text{Perimeter} = 2 \times 3 \text{ cm} + 9 \text{ cm} = 6 \text{ cm} + 9 \text{ cm} = 15 \text{ cm}
\]

For another type of triangle, the right triangle or right angled triangle, knowledge of two side lengths is sufficient to calculate the perimeter because of the special relationship that exists between the three sides. Recall that for a right triangle ABC, right angled at C as shown below, Pythagoras’ Theorem states:

\[
AC^2 + BC^2 = AB^2
\]

or

\[
a^2 + b^2 = c^2
\]

Thus, given the lengths of sides AC and BC it is possible to find the length of AC (the hypotenuse) and consequently the perimeter.
This is illustrated in the following example.

**Example 4**  In a triangle ABC, right-angled at B, side AB = 3 cm and side BC = 5 cm. Find the length of AC.

\[
AC^2 = AB^2 + BC^2
\]
\[
= 3^2 + 5^2
\]
\[
= 9 + 25
\]
\[
= 34
\]

This means that \( AC = \sqrt{34} \) or 5.83 (correct to 2 d.p.)

It is now possible to find the perimeter of the triangle ABC.

\[
P = AB + BC + AC
\]
\[
= 3 + 5 + 5.83 = 13.83 \text{ (correct to 2 d.p.)}
\]

Thus the perimeter of this triangle is 13.83 cm (correct to 2 d.p.).

In this example we were able to calculate the length of the longest side (the hypotenuse) in the triangle. However, sometimes you may be given the lengths of the hypotenuse and one side and asked to find the length of the remaining side.

**Example 5**  In the triangle ABC, right-angled at B, side AC = 13 cm and side AB = 5 cm. Find the length of BC and the perimeter of the triangle.

\[
AC^2 = AB^2 + BC^2
\]
\[
13^2 = 5^2 + BC^2
\]
\[
169 - 25 = BC^2 \quad \text{(Note: } BC^2 = 169 - 25 \text{ NOT } 169 + 25)\]
\[
144 = BC^2
\]
\[
\sqrt{144} = BC
\]
\[
12 = BC
\]

Now \( P = AB + BC + AC \)
\[
= 5 + 12 + 13 = 30
\]

Thus the perimeter of this triangle is 30 cm.
Activity 3.6

1. Calculate the perimeter of a triangle with sides 1.2 m, 0.9 m and 1.5 m.

2. Determine the perimeter of an isosceles triangle where the two equal sides have a length of 10.6 cm and the third side is 9.8 cm.

3. Calculate the perimeter of this figure made up of a rectangle with a triangle at each end. (Remember the perimeter is the distance around the outside of the figure as a whole.)

![Diagram of a figure made up of a rectangle with a triangle at each end.]

4. A rectangular plate measures 25 cm by 20 cm. From a corner, a triangular piece is removed with measurements as shown. Calculate the perimeter of the final shape.

![Diagram of a rectangular plate with a triangular piece removed.]

5. A garden bed 8 m by 6 m has a brick paved path 1 metre wide around it. Determine the outside perimeter of the path.

6. Calculate the perimeters of the following triangles. Use 2 decimal place accuracy if rounding is required.

   a) ![Diagram of a triangle with sides 12 mm, 10 mm, and 5 mm.]

   b) ![Diagram of a triangle with sides 17 mm, 20 mm, and 10 mm.]

Check your answers at the end of this Section.
3.7 Circumference

The perimeter of a circle is called its **circumference**, generally abbreviated to \( C \).

If a great number of circles were drawn, and their circumferences measured, it would be found in every case, that the circumferences were approximately 3.14 times the diameter \( (d) \). This irrational number cannot be found exactly so we refer to it as \( \pi \) standing for the Greek letter 'pi'. Accurately to 2 decimal places its value is 3.14. As a general rule, you should use the value programmed into your calculator unless instructed otherwise.

Thus the formula for finding the circumference of a circle is:

\[
C = \pi d
\]

**Example 1**  
Find, correct to 3 significant figures, the circumference of a circle if the radius is 3.5 mm.

Here \( d = 2 \times 3.5 \text{ mm} = 7 \text{ mm} \)

Consequently \( C = \pi \times d \)

\[
= 7\pi
\]

\[
= 21.99114...
\]

(Using the calculator)

Rounded correctly and inserting units, the circumference is \( 22.0 \text{ mm} \).

Sometimes we have to calculate a diameter given the circumference such as in this example.

**Example 2**  
To the nearest mm, what is the diameter of a metric trundle wheel that measures exactly 1m with each click?

Here \( C = 1 \text{ m} = 1000 \text{ mm} \).

Now \( C = \pi \times d \) or \( d = \frac{C}{\pi} \) after dividing both sides by \( \pi \).

Thus \( d = \frac{1000}{\pi} = 318.309... \)

Hence the diameter is 318 mm to the nearest mm.
Activity 3.7

Express your answers correct to two decimal places unless indicated otherwise.
Use $\pi = 3.142$.

1. Calculate the circumference of a circle with radius:
   a) 3.5 mm   b) 13.8 m   c) 4.2 cm

2. Calculate the circumference of a circle with diameter:
   a) 34.3 mm   b) 18.54 cm   c) 195.2 m

3. Determine the perimeter of a semi-circular region with radius 12.7 mm.

4. A circular race track 5 m wide is to have an inner and outer fence.
   Use $\pi = 3.142$ and give your answers correct to 1 decimal place.
   The radius of the inner circle of the track is 120 m.
   a) Calculate the length of the inner fence.
   b) Calculate the length of the outer fence.
   c) If the inner fence costs $12 per metre and the outer fence $8 per metre, determine the total cost of fencing the track.

5. The shape shown below is constructed by removing four quarter-circle regions each with radius 15 cm from the corners of a square with side length 50 cm.
   Calculate the perimeter of the shaded region. Use $\pi = 3.142$ and give your answer correct to 2 decimal places.

[Diagram]

Check your answers at the end of this Section.
3.8 The perimeter of composite figures

It may happen that in certain cases we are required to calculate the perimeters of figures composed of two or more simple figures as in the following example.

Example  Calculate the perimeter of the following figure.

The perimeter comprises of one curved and three straight edges.

Straight edges: \( P_1 = 10 + 4 + 14 = 28 \text{ cm} \)

Curved edge: \( P_2 = \frac{1}{4} (2\pi \times 4) \approx 6.28 \text{ cm} \)

Total: \( P = 28 + 6.28 \approx 34.28 \text{ cm} \)

Activity 3.8

Find the perimeters of the following figures.

1. \[
\begin{array}{c}
\text{10 m} \\
\text{20 m} \\
\text{4 cm}
\end{array}
\]

2. \[
\begin{array}{c}
\text{12 m} \\
\text{16 m}
\end{array}
\]

Check your answers at the end of this Section.
3.9 Area of a rectangle

The area of a figure is the amount of surface it covers.

This is a 1 centimetre square; it covers one square centimetre (1 cm²) of the surface of this page.

The formula for finding the area of a rectangle is:

\[ A_{\text{rectangle}} = B \times H \]

where
- \( A \) = number of square units in the area
- \( B \) = number of units in the base
- \( H \) = number of units in the height

Example

Find the area of a rectangle which is 5 cm along the base, and has a height of 30 mm.

\[ 30 \text{ mm} = 3 \text{ cm} \]

Hence \( A_{\text{rectangle}} = B \times H = 3 \text{ cm} \times 5 \text{ cm} = 15 \text{ cm}^2 \)

Activity 3.9

1. Find the area of a rectangle which is 2.5 m long and 0.5 m high.
2. Find the area of a rectangle which is 12 cm long and 0.5 mm high.
3. Find the area of a rectangular field which is 2 km by 1.5 km.
4. A square piece of metal has a side of 2.5 cm. Find its area.

Check your answers at the end of this Section.
3.10 Area of a triangle

\[ A_{\text{triangle}} = \frac{1}{2} (B \times H) \]

where \( A \) = number of square units in the area
\( B \) = number of units in the base
\( H \) = number of units in the height

Example 1 Find the area of a triangle with a base of 16 m and a height of 5 m.

\[ A_{\text{triangle}} = \frac{1}{2} (B \times H) \]
\[ = \left( \frac{1}{2} \times \frac{16}{1} \times \frac{5}{1} \right) \text{ m}^2 \]
\[ = 40 \text{ m}^2 \]

Example 2 Find the area of the triangle below.

Here the height needs to be calculated first using Pythagoras’ Theorem.
In fact \( h^2 = 13^2 - 12^2 = 169 - 144 = 25 \).

Then \( h = \sqrt{25} = 5 \) mm.

\[
A_{\text{triangle}} = \frac{1}{2} (B \times H) = \frac{1}{2} (12 \times 5) \text{ mm}^2 \\
= 30 \text{ mm}^2
\]

**Activity 3.10**

1. Calculate the area of a triangle with a base of 52 mm and a height of 36 mm.
2. Calculate the area of a triangle with a base of 22 mm and a height of 38 mm.
3. Calculate the area of a right triangle with a base of 40 mm which has a vertical side of length 28 mm.
4. Calculate the area of the triangle below.

Check your answers at the end of this Section.

### 3.11 Area of a circle

The formula for the area of a circle is: \( A_{\text{circle}} = \pi r^2 \)

Where \( \pi = 3.141592654\ldots \) (the number we met before) and \( r \) is the radius of the circle.

(Note: Strictly speaking, a circle has no area; we should speak of a circular region or the area enclosed by a circle.)

To see where the formula comes from, think of a circle split up into many segments as in the diagram below.
If these segments are rearranged as in the diagram below, we almost have a rectangular region.

![Diagram of segments rearranged into a rectangle](image)

The total lengths of these areas should be half the circumference of $2\pi r$

so $\frac{1}{2} 2\pi r = \pi r$.

The more segments we have, the closer our rearrangement will get to being a rectangle and we can use the formula for calculating the area of a rectangle: $A = BH$.

But $B$ is $\pi r$ and $H$ is $r$ so the formula becomes $A = \pi r \times r = \pi r^2$.

**Example** Find the area of a circle with a diameter of 7 m. Answer correct to 3 decimal places.

Since the diameter is 7 m, the radius is $\frac{7}{2} = 3.5$ m.

The $A = \pi r^2$

$= \pi \times 3.5 \times 3.5$

$= 38.485$ m$^2$.

**Activity 3.11**

Find the areas of the following circles. Use $\pi$ from the calculator and round to 2 decimal places.

1. Radius of 7 cm
2. Radius of 20 cm
3. Radius of 1 m
4. Radius of 4.2 m
5. Radius 21 m
6. Diameter of 200 km
7. Radius of 1.75 m

*Check your answers at the end of this Section.*
3.12 Area of composite figures

The areas of many complicated figures may be found by dividing them into figures whose area can be found easily, i.e. rectangle, circles, triangles etc.

Example  Find the area of the figure drawn.

We should treat the region as consisting of a rectangular region (in the middle) together with two semi-circular regions (one on each end of the rectangle).

The dimension of 70 m gives us the radius of our semi-circle. It can be seen that the diameter of these semi-circles (140 m) is the dimension for the ends of our rectangle. So we have:

\[
\text{Area of semi circle} = \frac{1}{2} A_{\text{circle}} = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi \times 70 \times 70 \text{ m}^2 = 7697 \text{ m}^2 \quad \text{(nearest m}^2)\]

\[
\text{Area of rectangle} = B \times H = (450 \times 140) \text{ m}^2 = 63000 \text{ m}^2
\]

\[
\therefore \text{ Total Area} = (7697 + 63000 + 7697) \text{ m}^2 = 78394 \text{ m}^2
\]
Activity 3.12
Find the area of the following composite figures:

1. 
   ![](image1)

2. 
   ![](image2)

3. 
   ![](image3)

4. 
   ![](image4)

Check your answers at the end of this Section.

3.13 Total surface area

A **prism** is a solid figure whose bases or ends have the same size and shape and are parallel to one another. Another way of saying this is that a prism has a uniform cross section. Examples are the rectangular prism, the triangular prism and the cylinder (which is a circular prism).

The **surface area** of a prism equals the sum of the areas of its faces. Because the top and bottom face have the same shape, the surface area can be always be found by means of the following formula:

\[
\text{Surface Area}_{\text{prism}} = 2 \times (\text{Area of Base}) + (\text{Perimeter of Base}) \times H
\]

Where \( H \) is the height of the prism.
The actual formula that needs to be used, depends on the shape of the base of the prism. For example, for a rectangular prism which is shaped like a shoebox, the area of the base is $L B$, while the perimeter is $2(L + B)$. Hence

$$\text{Surface Area \ rectangular prism} = 2(L \times B) + 2(L + B)H$$

This special formula can easily be derived by adding the six faces of the rectangular prism but the general formula works for all solids, including the cylinder.

**Example 1** Derive the formula for the surface area of a cylinder.

Here the area of the base is $\pi r^2$ and the perimeter is $2 \pi r$.

Hence:

$$\text{Surface Area \ cylinder} = 2 \pi r^2 + 2 \pi r h$$

Inserting brackets and extracting a common factor, this expression is frequently written as $2 \pi r (r + h)$.
Example 2 Calculate the total surface area of a triangular prism of height 10 cm and
with base in the shape of an equilateral triangle with side length 10 cm.
The perimeter of the base triangle is $3 \times 10 \text{ cm} = 30 \text{ cm}$
The area of the base triangle can be calculated if we know its height.
This can be obtained from Pythagoras’ Theorem.

\[ h = \sqrt{10^2 - 5^2} = \sqrt{75} \]

Thus the area = \[ \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 10 \times \sqrt{75} = 5 \sqrt{75} \]

Then surface area becomes
\[ 2 \times (\text{area of base}) + (\text{perimeter of base}) \times H = 2 \times 5\sqrt{75} + 30 \times 10 \]
\[ \approx 386.60 \text{ cm}^2 \]

Activity 3.13
Calculate the surface area of:
1. a cube with side length 4 cm
2. a rectangular prism with side lengths 3 m, 4 m, and 5 m
3. a cylinder with base radius 6 cm and height 9 cm
4. a prism of height 5 m and a base in the shape of a right triangle with
hypotenuse 5 m and one other side 3 m.

Check your answers at the end of this Section.
3.14 Volume of a prism

To measure a volume, it is necessary to consider 3 dimensional units. Such a unit is the one pictured below.

![Diagram of a cube](image)

This is a cube (a square block) whose sides are all one centimetre. Its volume is 1 cubic centimetre or 1 cm$^3$.

The size of the cubic unit can vary. It could be cm$^3$, m$^3$, km$^3$, etc.

Recall that a prism is a solid figure with a uniform cross section.

The volume of any prism can be found by finding the area of the bottom layer (base) of the solid, and multiplying this by the height. Expressing this in formula form:

$$V_{prism} = \text{area of base} \times \text{height}$$

For a rectangular prism the formula becomes

$$V_{\text{rectangular prism}} = L \times B \times H$$

since the base is a rectangle and $A (\text{rectangle}) = L \times B$
Example 1  Calculate the volume of a rectangular box whose dimensions are shown.

\[ V_{\text{rectangular prism}} = L \times B \times H \]

\[ = 3 \times 7 \times 11 \text{ cm}^3 \]

\[ = 231 \text{ cm}^3 \]

Activity 3.14
1. Calculate the volume of a rectangular box whose dimensions are shown:

2. Calculate the volume of a matchbox which measures 4 cm long, 2 cm wide and 1 cm high.

3. Calculate the amount of cement needed for a driveway which is 10 m long, 3 m wide and 0.1 m deep.

4. Calculate the volume of a box measuring 2.5 m long, 1.5 m wide and 0.5 m high.

Check your answers at the end of this Section.
3.15 Volume of a cylinder

The rule ‘Area of base \times height’ also applies to the cylinder. In particular:

\[ V_{\text{cylinder}} = \pi r^2 h \]

**Example** Calculate the volume of a cylindrical tank whose radius is 3 m and height 4 m.

\[ V(\text{cylinder}) = \pi r^2 h \]
\[ = (3.14 \times 3 \times 3 \times 4) \text{ m}^3 \]
\[ = 113.04 \text{ m}^3 \]
Activity 3.15
1. Calculate the volume of a cylinder with radius 8 cm and height 10 cm.
2. Calculate the volume of water necessary to fill a cylindrical tank with a diameter of 10 m to a depth of 4 m.
3. Calculate the volume of a pipe which is 3 m long with a diameter of 0.4 m.
4. Calculate the volume of a rod which has a radius of 0.2 cm and is 50 cm long.
5. A cylindrical can contains one litre of liquid. Given that its internal diameter is 80 mm, find the depth of liquid in the can, in millimetres.
6. A drum contains a roll of copper wire 15 mm in diameter and 1 km long. Calculate:
   (i) the volume of wire in (m$^3$) on the drum
   (ii) the cost of the drum of wire if copper costs $23,941 per m$^3$.

Check your answers at the end of this section.

3.16 Volume of a pyramid
A regular pyramid is a solid with a base (which may be any regular shape such as a triangle, square, rectangle, circle, etc.) and sides which meet at a point, called the vertex.

Some examples are:

(i) triangular pyramid (ii) rectangular pyramid (iii) circular pyramid cone

In general, the volume of a pyramid is given by:

$$V_{\text{pyramid}} = \frac{1}{3} \text{ area of base } \times h$$

That is, find the area of the base, divide by 3 and multiply by the vertical height.
Example 1  Calculate the volume of a pyramid with a base which measures 30 m by 20 m (rectangular) and whose vertical height is 15 m.

Use $V = \frac{1}{3} \text{(area of base)} \times \text{vertical height}$.

$\text{Area of base} = B \times L$

$= (30 \times 20) \text{ m}^2$

$= 600 \text{ m}^2$

$V = \frac{1}{3} \text{ area of base} \times h$

$= \frac{600}{3} \times 15 \text{ m}^3$

$= 3000 \text{ m}^3$

Example 2  Calculate the volume of a cone with a base radius of 5 cm and vertical height of 20 cm. Use $\pi = 3.14$.

$\text{Area of base} = \pi r^2$

$= \pi \times 5 \times 5 \text{ cm}^2$

$= 78.539816... \text{ cm}^2$

$V = \frac{1}{3} \text{(area of base)} \times h$

$= \frac{78.539816...}{3} \times 20 \text{ cm}^3$

$= 523.6 \text{ cm}^3$
Activity 3.16
1. Find the volume of a square pyramid if the base has sides of length 3.2 m and the height is 7.3 m.
2. Find the approximate volume of a cone (circular pyramid) if the radius of the base is 3 m and the height is 4 m.
3. Find the volume of a rectangular pyramid if the base measures 8.2 cm by 6.7 cm and its vertical height is 12 cm.
4. Sand is poured onto a circular area of radius 6 m, to a height of 14 m. What is the approximate volume of sand?

Check your answers at the end of this Section.

3.17 Volume of composite figures

Industrial equipment often comprises a combination of the objects that we have studied. In order to calculate the volume enclosed by a composite object it is necessary to calculate the volume of each component and then add the individual volumes.

Example A hopper used in an industrial process consists of cylinder beneath which is a cone. The cylinder has diameter 2 m and is 3 m deep. The cone is 1 m deep. Calculate the volume enclosed by the hopper.

Cylinder: \[ V_1 = \pi r^2 h = \pi (1)^2 \cdot 3 \approx 9.42 \text{ m}^3 \]
Cone: \[ V_2 = \frac{1}{3} \pi r^2 h = \pi (1)^2 \cdot 1 \approx 1.05 \text{ m}^3 \]
Total: \[ V = V_1 + V_2 \approx 9.42 + 1.05 \approx 10.47 \text{ m}^3 \]
Activity 3.17

1. The cross section of this right prism consists of a semicircle surmounted by an isosceles right angled triangle. Find:
   a) the length of AB, in millimetres
   b) the area of $\triangle ABD$, in square millimetres
   c) the area of the semicircle, in square millimetres
   d) the volume of the prism, in cubic metres.

2. Find the volume of this container, in litres. (Note: $1\text{ L} = 1000\text{ cm}^3$)

3. Find the volume of the frustum of the cone in the diagram adjacent.

(A frustum is a cone with the top section removed.) The original height of the cone was 0.33 m.

Note: all dimensions in metres.
4. A cylindrical water pipe of length 5 m has an external diameter of 50 mm and an internal diameter of 40 mm. Find:

a) the volume of water required to fill the pipe, in litres
b) the volume of metal present in the pipe, in cubic centimetres.

Check your answers at the end of this Section.
Assessment 3

1. Draw a diagram to illustrate the meaning of a ray.
2. Measure the longest line segment.

M N T

3. Illustrate the meaning of alternate angles.
4. Simplify $83^\circ 40' 19'' - 52^\circ 23' 42''$.
5. Give the supplement of $124^\circ 39' 40''$.
6. Change 2.456 radians to degrees, minutes and seconds.
7. Find the size of angle $g$.

8. Calculate the perimeter of a rectangle with sides of length 178 mm and 1.3 cm.
9. Calculate the perimeter of a circle of diameter 53 mm.
10. Triangle ABC is right angled at C. Side AB has length 15.2 cm and side AC has length 9.3 cm. Calculate the perimeter of the triangle to 3 significant figures.
11. Calculate the area of a rectangle with length 23 mm and width 5.7 cm to 3 significant figures.
12. A washer is such that its internal diameter is 10 mm and its external diameter is 30 mm. Calculate the area of the washer to 4 significant figures.
13. A rectangular pyramid has base dimensions 23 mm by 17 mm, with perpendicular height 35 mm. Calculate its volume to 3 significant figures.
14. Calculate the volume of a cylinder of diameter 16.2 cm and perpendicular height 15.5 cm, to 4 significant figures.
15. A food container for animals is of constant cross section and consists of a rectangle below which is a semicircle. If the dimensions of the rectangle are 40 cm by 12 cm, and the semicircle sits below the longer side of the rectangle, calculate the volume of the container to 4 significant figures if its length is 2 m.
Answers to activities

Activity 3.1

1. a) $\overrightarrow{AB}$, $\overrightarrow{BD}$, $\overrightarrow{AD}$, $\overrightarrow{BA}$, $\overrightarrow{DA}$, $\overrightarrow{DB}$
   b) $\approx 7$ cm
   c) $\overrightarrow{AB}$, $\overrightarrow{BD}$, $\overrightarrow{AD}$

2. $\angle CAB$ or $\angle A$

3. $\angle CXB$ and $\angle BXD$ or
   $\angle BXD$ and $\angle DXA$ or
   $\angle DXA$ and $\angle AXC$ or
   $\angle AXC$ and $\angle CXB$

Activity 3.2

1. a) $120^\circ$
   b) $37^\circ$
   c) $90^\circ$
   d) $152^\circ$
   e) $180^\circ$

2. a) $20^\circ$
   b) $47^\circ$
   c) $79^\circ 30'$
   d) $37^\circ 37'$
   e) $9^\circ 47' 30"$
   f) $53^\circ 33' 36"$

3. a) $70^\circ$
   b) $147^\circ$
   c) $44^\circ 30'$
   d) $96^\circ 13'$
   e) $79^\circ 9' 18"$
   f) $169^\circ 40' 27"$

4. a) $171^\circ 48'$
   b) $90^\circ$
   c) $136^\circ 8'$
   d) $66^\circ 46' 56"$
   e) $6^\circ 55'$
   f) $86^\circ 29' 18"$

5. a) $89^\circ 11'$
   b) $129^\circ 34'$
   c) $65^\circ 58' 4"$
   d) $320^\circ 45' 3"$
Activity 3.3

1. a) $0.5934^R$  
   b) $2.5449^R$  
   c) $0.2135^R$  
   d) $0.4095^R$  
   e) $2.5345^R$  
   f) $0.9819^R$

2. a) $114° \ 35' \ 30''$  
   b) $234° \ 54' \ 46''$  
   c) $164° \ 48' \ 41''$

3. 6.14 cm

4. 13.09 cm$^2$

Activity 3.4

1. a) $\angle BOD$ or $\angle DOB$  
   b) $\angle AOC$, $\angle BOD$ or $\angle COA$, $\angle DOB$

2. a) $41°$  
   b) $83°$  
   c) $a° = 165°$, $b° = 15°$, $c° = 165°$  
   d) $x° = 27°$  
   e) $x° = 35°$  
   f) $y° = 70°$  
   g) $r° = 100°$
   h) $a° = 70°$, $b° = 70°$, $c° = 40°$  
   i) $x° = 40°$, $y° = 55°$  
   j) $x° = 72°$  
   k) $x° = 65°$, $y° = 115°$, $z° = 65°$  
   l) $x° = 67°$  
   m) $p° = 101°$
   n) $t° = 27°$

Activity 3.5

1. 16 mm
2. 12 cm
3. 2480 cm
4. 10.8 mm
5. 412 cm

Activity 3.6

1. 3.6 m
2. 31 cm
3. 66 mm
4. 86.60 cm
5. 36 m
6. a) 30 m  
   b) 47.54 mm
Activity 3.7
1. a) 21.99 mm  
b) 86.72 m  
c) 26.39 cm

2. a) 107.77 mm  
b) 58.25 cm  
c) 613.32 m

3. 65.30 mm

4. a) 754.1 m  
b) 785.5 m  
c) $15,333.20

5. 174.26 cm

Activity 3.8
1. 65.71 m
2. 71.98 cm

Activity 3.9
1. 1.25 m²
2. \(60 \text{ mm}^2 = 0.6 \text{ cm}^2\)
3. 3 km²
4. 6.25 cm²

Activity 3.10
1. 936 mm²
2. 418 cm²
3. 560 mm²
4. 48 cm²

Activity 3.11
1. 153.94 cm²
2. 1256.64 cm²
3. 3.14 m²
4. 55.42 m²
5. 1385.44 m²
6. 31,415.93 km²
7. 9.62 m²
Activity 3.12
1. 72 mm²
2. 84 m²
3. 21.99 m²
4. 189.27 m²

Activity 3.13
1. 96 cm²
2. 94 m²
3. 565.5 cm²
4. 72 m²

Activity 3.14
1. 24 m³
2. 8 cm³
3. 3 m³
4. 1.875 m³

Activity 3.15
1. 2009.6 cm²
2. 314 m³
3. 0.3768 m³
4. 6.28 cm³
5. 199 mm
6. i) 0.1767 m³
   ii) $4230.72

Activity 3.16
1. 24.917 m³
2. 37.68 m³
3. 219.76 cm³
4. 528 m³

Activity 3.17
1. a) 70.7 mm
   b) 2500 mm²
   c) 3926.99 mm²
   d) 0.0553 m³
2. 28 L
3. 0.014 m³
4. a) 6.3 L
   b) 3534.3 cm³
Section 4 – Trigonometry

4.1 Basic trigonometric ratios

If triangles are similar:

- Their corresponding angles are congruent (are the same size).
- The ratios of corresponding sides are equal.

In the diagram below there are several similar right triangles all containing the angle $A$.

$\triangle ABC \sim \triangle APQ \sim \triangle AXY \sim \triangle AMN$

(The symbol $\sim$ means ‘similar to’.)

So the ratios between corresponding sides will be the same for each triangle.

$$\frac{BC}{AB} = \frac{PQ}{AP} = \frac{XY}{AX} = \frac{MN}{AM} \quad \text{for example}$$

or

$$\frac{AC}{BC} = \frac{AQ}{PQ} = \frac{AY}{XY} = \frac{AN}{MN}$$

Because these ratios are the same, no matter which triangle they are in, we can use the ratio to determine the size of angle $A$. To do this, we must know the different ratios of sides for each size that $A$ could have.

There are three ratios that we are particularly interested in and which we will give special names.
For an angle in a right triangle we refer to the sides as **opposite** (to the angle), **adjacent** (to the angle) and the **hypotenuse** (opposite the right angle).

For angle \( A \) we define the following ratios. They will be the same value no matter what the size of the triangle that \( A \) is in, provided that it is a right triangle (some other angle, not \( A \), has a size of 90°).

The **sine** of angle \( A \) or \( \sin A \) = \[ \frac{\text{length of the opposite side}}{\text{length of the hypotenuse}} \]

The **cosine** of angle \( A \) or \( \cos A \) = \[ \frac{\text{length of the adjacent side}}{\text{length of the hypotenuse}} \]

The **tangent** of angle \( A \) or \( \tan A \) = \[ \frac{\text{length of the opposite side}}{\text{length of the adjacent side}} \]

Many people remember these ratios using the mnemonic **SOHCAHTOA** for sine (S) is opposite (O) over hypotenuse (H), cosine (C) is adjacent (A) over hypotenuse (H) and tangent (T) is opposite (O) over adjacent (A).

To find each of these ratios for an angle of, for example, size 30° we can draw a right-angled triangle with one angle 30° and measure the three sides for the particular triangle we have chosen. For the triangle below, the sides measure 4 cm, 6.9 cm and 8 cm.

We can now calculate the ratios:

\[
\sin 30° = \frac{4}{8} = 0.5 \\
\cos 30° = \frac{6.9}{8} = 0.86 \\
\tan 30° = \frac{4}{6.9} = 0.58
\]
You should satisfy yourself that if you draw another right angled triangle with an angle of 30° but with different length sides that these ratios have the same values as we calculated above.

Rather than calculating the sin, cos and tan of every possible angle size using this method, we use the calculator to determine this ratio.

Make sure your calculator is in degree mode. Check the calculator handbook to find out how to do this. Then press the ‘SIN’ button followed by 30 and sin 30 should appear on the screen. Then press the ‘enter’ or ‘exe’ button and the value of sin 30° will be shown. You can repeat this process to find out the true values of cos 30° and tan 30° by using the COS and TAN buttons respectively.

You should find the following values:

\[
\sin 30° = 0.5 \quad \cos 30° = 0.8660 \quad \tan 30° = 0.5774
\]

Sometimes it is necessary to find a reciprocal or a power of a trigonometric ratio. For powers a special notation needs to be used to avoid confusion. For example, to obtain the square of sin 30°, we cannot write sin 30°² as we would in \(x^2\). The notation used is \(\sin^2 30°\). Note that this notation is not recognised by elementary scientific calculators and so, if we wish to calculate \(\sin^2 30°\) on such a calculator, we should enter the expression as \((\sin 30°)^2\). This calculation can be done in stages by first obtaining \(\sin 30°\) and then using the square key.

Consider these examples.

**Example 1** Calculate \(\cos^2 76°\) correct to four decimal places.

\[
\cos^2 76° = (\cos 76°)^2
\]

\[
= (0.24192\ldots)^2
\]

\[
= 0.0585262\ldots
\]

\[
= 0.0585 \text{ correct to 4 dp}
\]

**Example 2** Calculate \(1/\sin (15° 12´)\) correct to four decimal places.

\[
1/\sin (15° 12´) = 1/0.262189\ldots
\]

\[
= 3.814039\ldots
\]

\[
= 3.8140 \text{ correct to 4 dp}
\]
Activity 4.1

1. Use a calculator to determine the tan, sin and cos for each of the angles indicated and hence complete the table below. Give results correct to 4 decimal places.

<table>
<thead>
<tr>
<th>A</th>
<th>sin A</th>
<th>cos A</th>
<th>tan A</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>53.2°</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>68° 15’ 20”</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Calculate the following correct to 4 decimal places:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>sin 21° + cos 43°</td>
<td>b)</td>
<td>sin² 47°</td>
</tr>
<tr>
<td>c)</td>
<td>cos(tan 823°)°</td>
<td>d)</td>
<td>1/tan 43°</td>
</tr>
<tr>
<td>e)</td>
<td>cos² 76° 12’ 55”</td>
<td>f)</td>
<td>sin² 15° + cos² 15°</td>
</tr>
</tbody>
</table>

Check your answers at the end of this Section.
4.2 Exact values

When we calculated the values of \( \sin 30^\circ \), \( \cos 30^\circ \) and \( \tan 30^\circ \), you noticed that we had to round our answers for \( \cos 30^\circ \) (= 0.8660) and \( \tan 30^\circ \) (= 0.5774) but that the value of \( \sin 30^\circ \) was exact. In fact \( \sin 30^\circ = 0.5 \).

Using Pythagoras’ Theorem and two special triangles, exact values can easily be obtained. In the following example the exact values of \( \sin 45^\circ \), \( \cos 45^\circ \) and \( \tan 45^\circ \), are obtained. In the activity you are asked to give exact values of \( \sin 30^\circ \), \( \sin 60^\circ \), \( \cos 30^\circ \), \( \cos 60^\circ \), \( \tan 30^\circ \) and \( \tan 60^\circ \).

**Example** Obtain the exact values of \( \sin 45^\circ \), \( \cos 45^\circ \), and \( \tan 45^\circ \)

We start by drawing an isosceles right angled triangle. If we make the shorter sides each 1 unit long, then using Pythagoras’ Theorem, the hypotenuse can be calculated.

\[
\text{In } \triangle ABC \quad b^2 = a^2 + c^2 \\
= 1^2 + 1^2 \\
= 2
\]

Hence \( b = \sqrt{2} \).

Now we can obtain the ratios for an angle of 45° without a calculator because for angle \( A \), which is 45°, we know that the length of the side opposite is 1, the length of adjacent side is 1 and the length of the hypotenuse is \( \sqrt{2} \).

Then:

\[
\sin 45^\circ = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos 45^\circ = \frac{1}{\sqrt{2}} \quad \text{and} \quad \tan 45^\circ = \frac{1}{1}
\]

Here the expression \( \frac{1}{\sqrt{2}} \) is multiplied by \( \frac{\sqrt{2}}{\sqrt{2}} \) (which is 1) to obtain \( \frac{\sqrt{2}}{2} \).

Thus we write:

\[
\sin 45^\circ = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos 45^\circ = \frac{\sqrt{2}}{2}
\]
Activity 4.2

Draw an equilateral triangle with vertices A, B and C. Let each side be 2 units. Draw a perpendicular bisector from A meeting BC in D.

1. Noting that BD is 1 unit, calculate, using Pythagoras’ Theorem, the length of AD.
2. Use angle ABD (60°) to obtain the exact value of \( \sin 60°, \cos 60° \) and \( \tan 60° \).
3. Use angle BAD (30° because it is the bisected angle BAC) to obtain the exact value of \( \sin 30°, \cos 30° \) and \( \tan 30° \).

Check your answers at the end of this Section.

4.3 Trigonometric relationships and identities

In this topic we investigate some further properties of the trigonometric ratios in the right angled triangle. We shall refer to \( \triangle ABC \) drawn below for which the trigonometric ratios have been listed in terms of the labels that are generally given to the sides opposite the angles A, B and C.

\[
\begin{align*}
\sin A &= \frac{a}{b} \\
\cos A &= \frac{c}{b} \\
\tan A &= \frac{a}{c}
\end{align*}
\]

1. Maximum and minimum values of \( \sin, \cos \) and \( \tan \).

Imagine the size of angle A to be 0°. In that case the opposite side a will have no length ie \( a = 0 \). Further, the lengths of b and c will be the same ie \( c = b \).

This means that: \( \sin 0° = \frac{a}{b} = \frac{0}{b} = 0 \quad \cos 0° = \frac{c}{b} = \frac{b}{b} = 1 \quad \tan 0° = \frac{a}{c} = \frac{0}{c} = 0 \)

Now imagine the size of angle A to be 90°. In that case the adjacent side c will have no length ie \( c = 0 \). Further, the lengths of a and b will be the same ie \( a = b \).

This means that: \( \sin 90° = \frac{a}{b} = \frac{b}{b} = 1 \quad \cos 90° = \frac{c}{b} = \frac{0}{b} = 0 \quad \tan 90° = \frac{a}{c} = \frac{a}{0} \)

The ratio \( \frac{a}{0} \) cannot be simplified since division by zero is not defined (it produces an infinitely large number). Hence we say that \( \tan 90° \) is not defined.
Thus we obtain the following limiting values on sine, cosine and tangent in right-angled triangles:

\[ 0 \leq \sin \theta \leq 1 \quad 0 \leq \cos \theta \leq 1 \quad \tan \theta \geq 0 \]

Here we have used the Greek letter \( \theta \) (theta) to designate any angle. This is a common practice, especially in higher level trigonometry.

2. Relationship between sine and cosine

Recall from section 1.1 that when two angles have a sum that is a right angle ie 90° they are called **complementary** angles. The two acute angles in a right angled triangle are complementary.

![Diagram of right-angled triangle](image)

Here \( \angle X + \angle Z = 90° \)

Now \( \sin X = \frac{x}{y} \quad \text{and} \quad \cos X = \frac{z}{y} \)

Also \( \sin Z = \frac{z}{y} \quad \text{and} \quad \cos Z = \frac{x}{y} \)

We notice that the sine of one of the acute angles is always equal to the cosine of the other, complementary, angle. In fact this is the reason for the term **cosine**, meaning the **sine** of the **complementary** angle.

3. Trigonometric identities

**Identities** in Mathematics refer to an equality that remains true regardless of the values of any variable that appear within it. The symbol \( \equiv \) is sometimes used to indicate a mathematical identity. A typical example is \( a + 0 = a \) which is true, no matter the value of \( a \). An identity is different from an **equality** which is true under more particular conditions. For example \( \sin \theta = 1 \) is only true for \( \theta = 90° \).

There are two important trigonometric identities that can be easily proven.

They are:

\[ \frac{\sin \theta}{\cos \theta} = \tan \theta \quad \text{and} \quad \sin^2 \theta + \cos^2 \theta = 1 \]
Example 1  Use $\triangle ABC$ with $\angle B = 90^\circ$ to prove that $\frac{\sin A}{\cos A} = \tan A$.

Using $\triangle ABC$, we defined $\sin A = \frac{a}{b}$ and $\cos A = \frac{c}{b}$.

Then:

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{b}}{\frac{c}{b}} = \frac{a}{c} = \tan A$$

Example 2  In the right triangle $XYZ$, $\angle Y = 90^\circ$ and $\sin X = 0.6$.

What is the value of:

a) $\cos X$?  
b) $\tan X$?  
c) $\sin Z$?

a) Since $\sin^2 \theta + \cos^2 \theta = 1$, for angle $X$ we obtain $\sin^2 X + \cos^2 X = 1$ from which we obtain $\cos^2 X = 1 - \sin^2 X$.

Thus: $\cos^2 X = 1 - 0.6^2 = 1 - 0.36 = 0.64$

Then ignoring the negative square root we obtain

$$\cos X = \sqrt{0.64} = 0.8$$

$$\tan X = \frac{\sin X}{\cos X} = \frac{0.6}{0.8} = 0.75$$

b) Using the identity proven in Example 1

c) Since $Z$ is the complementary angle to angle $X$, $\sin Z = \cos X = 0.8$

Note:  Despite not knowing the actual lengths of the sides, a diagram would have been useful in this example. We could let the hypotenuse $y = 1$ with $x = 0.6$. Then, by Pythagoras’ Theorem, we would have obtained $z = 0.8$ from which the other results follow.
Activity 4.3

1. Prove \( \sin^2 A + \cos^2 A = 1 \) using \( \triangle ABC \) with \( \angle B = 90^\circ \).

2. If triangle PQR has \( \angle P = 90^\circ \) and \( \tan Q = \frac{5}{12} \), what is the value of:
   a) \( \sin Q \)?
   b) \( \cos Q \)?
   c) \( \tan R \)?

Check your answers at the end of this Section.

4.4 Reciprocal trigonometric ratios

Sometimes use is made of the so-called reciprocal ratios. They are defined as follows:

The **cosecant** of angle \( A \) or cosec \( A \) = \( \frac{\text{length of the hypotenuse}}{\text{length of the opposite side}} \)

The **secant** of angle \( A \) or sec \( A \) = \( \frac{\text{length of the hypotenuse}}{\text{length of the adjacent side}} \)

The **cotangent** of angle \( A \) or cot \( A \) = \( \frac{\text{length of the adjacent side}}{\text{length of the opposite side}} \)

The ratios are called reciprocal because they are the reciprocals (‘1 over’) of the ratios previously defined.

Thus: cosec \( A = \frac{1}{\sin A} \) \hspace{1cm} \text{sec} \( A = \frac{1}{\cos A} \) \hspace{1cm} \text{cot} \( A = \frac{1}{\tan A} \)

One identity that is easily proved is \( \cot \theta = \frac{\cos \theta}{\sin \theta} \)

Most calculators do not have special buttons for the reciprocal ratios and you need to proceed as in the examples below.

**Example 1** Calculate sec \( 34^\circ \) correct to 4 dp

\[
\text{sec} \ 34^\circ \ = \ \frac{1}{\cos 34^\circ} = \frac{1}{0.829037...} \\
= 1.2062 \ \text{correct to 4 dp}
\]
Example 2  What is the exact value of cosec 30°?

\[
cosec 30° = \frac{1}{\sin 30°} = \frac{1}{0.5} = 2
\]

Example 3  Show that \(1 + \tan^2 A = \sec^2 A\)

We know that \(\sin^2 A + \cos^2 A = 1\)

Dividing both sides by \(\cos^2 A\) produces:

\[
\frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A} = \frac{1}{\cos^2 A}
\]

\[\text{ie} \quad \left(\frac{\sin A}{\cos A}\right)^2 + 1 = \left(\frac{1}{\cos A}\right)^2\]

\[\text{ie} \quad \tan^2 A + 1 = \sec^2 A\]

or \(1 + \tan^2 A = \sec^2 A\)

Activity 4.4

1. Calculate correct to 4 decimal places the following:
   a) \(\sec 76°\)  b) \(\cosec 46.12°\)  c) \(\cot 76° 02' 55''\)

2. Investigate what the maximum and minimum values are of the reciprocal ratios in a right triangle.

3. What can \(\frac{1}{\sec \theta}\) be simplified to?

4. Using \(\triangle ABC\) with \(\angle B = 90°\), prove \(1 + \cot^2 A = \cosec^2 A\)

5. If triangle PQR has \(\angle P = 90°\) and \(\sin Q = \frac{3}{5}\), what is the value of:
   a) \(\sec Q\)  b) \(\cosec Q\)  c) \(\cot Q\)

Check your answers at the end of this Section.
4.5 Inverse trigonometric values

Knowing a trigonometric ratio, it is possible to work backwards and obtain the size of the angle.

For example, suppose we know that the sin of an angle works out to be 0.5 ie we know that \( \sin A = 0.5 \), what is the size of \( A \)?

On the calculator we have to perform an ‘inverse’ process which on most calculators is designated with a superscript \(-1\) such as in \( \sin^{-1} \).

It is accessed by using the shift or second function key followed by the particular trigonometric key.

\[
\text{SHIFT} \quad \sin^{-1} \quad \text{D} \quad \sin
\]

We write \( A = \sin^{-1} 0.5 \) and, provided your calculator is in **degree mode**, pressing the shift or second function key then sin followed by 0.5 will show \( \sin^{-1} 0.5 \) on your display. Then pressing the ‘exe’ key will produce the answer 30 for 30°.

Note that the inverse notation such as \( \sin^{-1} \) is actually quite confusing because in Mathematics the power \(-1\) usually designates ‘1 over’. Thus \( \sin^{-1} 30° \) could be interpreted as \( 1/\sin 30° \).

For this reason, some graphics calculators such as the HP series use the notation asin, acos and atan (the a is an abbreviation for ‘arc’). Thus on these calculators asin 0.5 means \( \sin^{-1} 0.5 \). Also note that some computer languages use the notation INVSIN, INVCOS and INVATAN. Thus we would write INVSIN 0.5 for \( \sin^{-1} 0.5 \).

**Example 1** If \( \sin T = 0.2662 \), determine the size of the acute angle \( T \), correct to the nearest degree.

Here \( T = \sin^{-1} 0.2662 = 15.43826\ldots \) = 15° to the nearest degree.

**Example 2** The cosine of an acute angle is 0.576. Calculate the size of the angle in degrees, minutes and seconds.

Let the angle be \( \theta \). Then \( \cos \theta = 0.576 \)

\[ \text{ie } \theta = \cos^{-1} 0.576 = 54.830307\ldots \]

\[ = 54° 49’ 49” \text{ as required.} \]

**Example 3** Determine the size of the acute angle \( X \) given that \( \sec X = 2 \).

If \( \sec X = 2 \) then \( \frac{1}{\cos X} = 2 \) \( \text{ie } \cos X = 0.5 \)

Thus \( X = \cos^{-1} 0.5 \) or \( X = 60° \)
Example 4  Solve the equation $2\tan A - 0.3 = 0.9$ correct to the nearest minute.

If $2\tan A - 0.3 = 0.9$ then $2\tan A = 1.2$ i.e. $\tan A = 0.6$

Then $A = \tan^{-1} 0.6$ or $A = 30.963756\ldots ^\circ = 30^\circ 58'$ to the nearest minute.

Activity 4.5

1. Complete the following table maintaining the same level of rounding in each column.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$\sin B$</th>
<th>$\cos B$</th>
<th>$\tan B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$47^\circ$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$0.495$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.62$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$0.0146$</td>
</tr>
</tbody>
</table>

2. If $\sin \theta = 0.3471$, determine $\theta$ correct to the nearest minute.

3. Calculate $\tan^{-1} 4.87$ correct to the nearest second.

4. Calculate $\cot^{-1} 2$ correct to 2 decimal places.

5. Solve the equation $1 - \cos X = 0.34$ to the nearest degree.

Check your answers at the end of this Section.
4.6 Finding an angle size

Because of the special relationship that exists between ratios of sides and angles in a right triangle, it is possible to find the size of an angle in such a triangle if we know two side lengths.

The following examples illustrate this.

Example 1  Determine, correct to the nearest degree, the size of angle \( P \) in the triangle shown below.

![Diagram of a triangle with sides labeled Q, P, and R, and angles at P and Q marked.]

In relation to angle \( P \), the lengths of the opposite side and hypotenuse are known. Hence we use the sine ratio (remember SOHCAHTOA).

\[
\sin P = \frac{QR}{PR} = \frac{36.8}{76.5} = 0.4810457\ldots
\]

Hence \( P = \sin^{-1} 0.4810457\ldots = 28.753724\ldots \)

Rounding correctly gives us \( P = 29^\circ \)

Example 2  Determine, correct to the nearest second, the size of the smallest angle in a 3, 4, 5 triangle.

The smallest angle will be opposite the smallest side. Let this angle be \( \theta \).

Then \( \tan \theta = \frac{3}{4} = 0.75 \)

Consequently \( \theta = \tan^{-1} 0.75 \)

\( = 6.869897\ldots^\circ \)

\( = 36^\circ 52'12'' \)
Activity 4.6
1. To the nearest degree, find the size of angle $Y$ in each case.

   a) $Y$
   
   b) $Y$

2. Find, correct to the nearest minute, the size of the largest acute angle in a 5, 12, 13 triangle.

Check your answers at the end of this Section.

4.7 Solving a right triangle

To solve a triangle simply means to use the information we have about the sizes of its angles and lengths of its sides, in order to calculate the sizes of the remaining angles and lengths of sides. Of course, in a right triangle, one angle is already known which means that the other two must be complementary.

Example 1 Solve the triangle expressing your answers with an accuracy of two decimal places.

Angles $A$ and $C$ must be complementary so:

$A = 90 - 25 = 65^\circ$

Now to find the length $b$ of the hypotenuse $\overline{AC}$ we need to use trigonometry.
We can say that \( \sin 25^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{10}{b} \)

The calculator will give us a value for \( \sin 25^\circ \) which is 0.422612...

So we have the equation \( 0.4226... = \frac{10}{b} \)

We solve for \( b \):

\[
b = \frac{10}{0.4226...} = 23.66 \text{m (2 d.p.)}
\]

To find the length of side \( a \), the only other side whose length is not known, we could use Pythagoras’ Theorem or we could use trigonometry.

Using Pythagoras:

\[
a^2 + c^2 = b^2
\]

\[
a^2 + 10^2 = (23.66)^2
\]

\[
a^2 = (23.66)^2 - 10^2
\]

\[
a^2 = 559.8 - 100
\]

\[
a = 21.44 \text{m}
\]

Using trigonometry:

\[
\tan 25^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{10}{a}
\]

\[
\tan 25^\circ = 0.4463
\]

so \( a = \frac{10}{0.4463} = 21.45 \text{m} \)

The slight discrepancy between the above results is due to rounding errors.

In some cases we need to use inverse trigonometric buttons on the calculator.

**Example 2** Solve this triangle, accurate to two decimal places.

\[PQ = 12 \text{cm} \]

\[PR = 18 \text{cm} \]

Pythagoras’ Theorem will give us the third side.

\[
r^2 = q^2 + p^2
\]

\[
r^2 = 18^2 + 12^2
\]

\[
r^2 = 324 + 144 = 468
\]

\[
r = 21.63 \text{cm (2 d.p.)}
\]
Section 4 Trigonometry

To determine the size of $\angle P$ or $\angle Q$ we must use trigonometry.

\[
\tan P = \frac{\text{opposite}}{\text{adjacent}} = \frac{12 \text{ cm}}{18 \text{ cm}} = 0.6667 \text{ (4 d.p.)}
\]

so \[ P = \tan^{-1} 0.6667 = 33.69^\circ \text{ (2 d.p.)} \]

Hence \[ Q = 90^\circ - 33.69^\circ = 56.31^\circ \]

**Activity 4.7**

Solve each of these triangles. For rounding, if required, use one decimal place.

1. $\triangle PQR$ where $P = 90^\circ$, $A = 60^\circ$, and $b = 29 \text{ m}$

2. $\triangle XYZ$ where $Y = 90^\circ$, $A = 54^\circ$, and $b = 17 \text{ cm}$

3. $\triangle DEF$ where $D = 45^\circ$, $2 \text{ cm}$, and $E = 45^\circ$

4. $\triangle ABC$ where $B = 90^\circ$, $A = 60^\circ$, and $b = 3 \text{ m}$

Check your answers at the end of this Section.
4.8 Practical right triangle problems

Following is a typical applied problem encountered in Mechanics.

**Example 1** If a force of 54 newtons is applied at an angle of 27° to the horizontal, how much force acts in the vertical direction? Use 1 decimal place accuracy.

Let the vertical force be \( x \) newtons.

Then the triangle of forces is as shown in the diagram.

We obtain

\[
\sin 27^\circ = \frac{x}{54} \quad \text{or} \quad x = 54 \sin 27^\circ
\]

Hence \( x = 24.5 \) N.

The following is a typical Electronic Engineering application problem.

**Example 2** For an alternating current circuit, the impedance \( Z \) can be represented as the hypotenuse of a right triangle, the resistance \( R \) is represented horizontally and the inductive reactance \( X_L \) is represented vertically.

The angle between \( R \) and \( Z \) is the phase angle.

If \( R = 34 \) Ω and \( X_L = 25 \) Ω,

find \( Z \) and the phase angle. Accurate to two decimal places.

Using Pythagoras’ Theorem:

\[
Z^2 = R^2 + X_L^2
\]

\[
Z^2 = 34^2 + 25^2
\]

\[
Z^2 = 1781
\]

So the impedance \( Z = 42.20 \) Ω

Now \( \tan \theta = \frac{25}{34} \) so \( \theta = \tan^{-1} \frac{25}{34} = 36.33^\circ \)

Hence the phase angle is 36.33°.
The following problem is frequently encountered in the building industry.

**Example 3**  The pitch of a roof is defined as the ratio of the rise over the horizontal distance. If a building has a roof with pitch 4:12, what angle does the roof make with the horizontal? If the house is 19 m wide and the roof is symmetrical, how long is the roof?

Let the roof be \( \ell \) metres long. From the diagram

\[
\tan \theta = \frac{4}{12}
\]

\[
\theta = \tan^{-1} \left( \frac{4}{12} \right)
\]

\[
\theta = 18.43^\circ
\]

Then

\[
\cos \theta = \frac{19/2}{\ell}
\]

\[
\ell = \frac{9.5}{\cos 18.43^\circ} = 10.01 \text{ m}
\]

Thus the roof is 10 m long.

**Activity 4.8**

Solve the following problems. Use 2 decimal places for both sides and angles.

1. A force is applied so that the horizontal component is 23 N and the vertical component is 16 N. At what angle is the force applied? What is the total force?

2. In an alternating current circuit, the impedance is 45 ohms, the phase angle is 60°. Calculate the resistance, \( R \), and the inductive reactance \( X_L \).

3. A ladder leans against the side of a building and makes an angle of 70° with the ground. The foot of the ladder is 3 metres from the building. Find:
   a) how high up on the building the ladder reaches
   b) the length of the ladder.

4. A plane rises from a take-off and flies at a fixed angle of 9° with the horizontal ground. When it has gained 400 m in altitude, find, to the nearest 10 m, the horizontal distance flown.

5. If the base angle of an isosceles triangle is 28° and each leg is 45 cm, find, to the nearest cm, the length of the base.

6. In a triangle, an angle of 50° is included between sides of 12 and 18. Find the altitude to the side of length 12.

7. Find the sides of a rectangle to the nearest cm if a diagonal of 24 cm makes an angle of 42° with a side.
8. A rhombus has an angle of 76° and a long diagonal of 40 cm. Find the length of the short diagonal.

9. A road is inclined uniformly at an angle of 6° with the horizontal. After driving 10 000 m along this road find the:
   a) increase in the altitude of the driver
   b) horizontal distance that has been driven.

10. An aeroplane travels 15 000 m through the air at a uniform angle of climb, thereby gaining 1 900 m in altitude. Find its angle of climb.

11. Tangents $PA$ and $PB$ are drawn to a circle from external point $P$. $\angle APB = 40°$ and $PA = 25$ m. Find the radius of the circle.

12. A concrete channel is to be built to measure 15 m across the top, 9 m across the bottom and to have uniform depth of 8 m. If one side is be inclined at 12° to the vertical, what must be the inclination to the vertical at the other side?

Check your answers at the back of this Section.

4.9 Angle of elevation and depression

Many trigonometry problems involve the terms angle of elevation and angle of depression.

An angle of elevation occurs where an observer has to raise or elevate the eyes to view something – the top of a tree, or building for example.

An angle of depression occurs when an observer lowers the eyes to view an object such as a boat out at sea or a car in the street below a building.

Note: The angles are made with the horizontal.
Example To the nearest metre, how high is a tree where the angle of elevation at 20 m from the base, is 35°?

Let the height of the tree be \( t \) metres.

Then \( \tan 35^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{t}{20} \)

Now \( \tan 35^\circ = 0.7002… \)

so \( \frac{t}{20} = 0.7002 \)

and \( t = 20(0.7002) = 14.004 \)

So the height of the tree is 14 m.

Activity 4.9

Solve the following problems. Use 2 decimal places for both sides and angles.

1. A surveyor sights the top of a TV mast from a point 42 m from its base. The angle of elevation is 43.2°. How high is the mast?

2. Sighting the top of a monument, William found the angle of elevation to be 16°. The ground is level and his eyes are 1.7 m above ground. If the monument is 86 m high, find the distance from William to the foot of the monument.

3. Find to the nearest degree the angle of elevation of the sun when a tree 60 m high casts a shadow 10 m in length.

4. At a certain time of day, the angle of elevation of the sun is 34°. Find the shadow cast by a building 70 m high.

5. A light at C is projected vertically to a cloud at B. An observer at A, 1000 m horizontally from C, notes the angle of elevation of B. Find the height of the cloud if \( \angle A = 37^\circ \).

6. A lighthouse built at sea level is 180 m high. From its top, the angle of depression of a buoy is 24°. Find the distance from the buoy to the foot of the lighthouse.

7. An observer on the top of a hill 300 m above the level of a lake, sighted two ships directly in line. Find, to the nearest metre, the distance between the boats if the angles of depression noted by the observer were 20° and 15°.

8. The angle of depression of a yacht at S from the top of a 50 m cliff, C, is 15°. The yacht sails away from the cliff on a course perpendicular to the cliff to a point T. The angle of depression of the yacht from C is now 10°. If the yacht took 20 seconds to sail from S to T, what was her speed in kilometres per hour?

Check your answers at the end of this Section.
4.10 Compass bearings

There are a number of ways of taking readings from a compass. We will investigate three of them.

1. By points on the compass

The cardinal directions on a compass are the North, South, East and West, usually abbreviated to N, S, E and W.

At 45° these directions are NE, SE, SW and NW.

It is also possible to name the points of the compass between these. For example, half way between S and SW is SSW. This is found easily by starting at the first letter’s position (S) and then travelling to the next pair of letters’ position (SW but stopping half way).
2. Using cardinal points and angle sizes

This is best understood by example.

**Example 1** Draw W 39° N.

We start due west, then rotate 39° towards the north.

\[ \text{Note: } W 39° \text{ N could also be written as N 51° W.} \]

3. Using bearings

This method relies solely on numbers, measuring from due north and rotating clockwise around the compass.

Due north is taken as 0°, due east is 090° (ie 90°), due south as 180° and due west as 270°.

\[ \text{Note: } \]
Example 1  Illustrate a bearing of 204°

Activity 4.10
It is not essential to use a protractor for your diagrams. However, make sure your answers are clear and tidy.

1. Illustrate by diagram the following points of the compass:
   a) NNE       b) WSW       c) ESE       d) SSE
2. Illustrate by diagram the following compass directions and give their alternative forms:
   eg N 31° E for E 59° N.
   a) N 68° W    b) W 10° S    c) E 17° S    d) S 75° W
3. Illustrate by diagram the following bearings:
   a) 062°       b) 305°       c) 125°       d) 225°

Check your answers at the end of this Section.
4.11 Applied bearing problems

Trigonometry is frequently applied to problems involving bearings. To solve these problems, it is convenient to know that if the bearing from A to B is known, then the bearing from B to A can be found by either adding 180° to the bearing from A to B, or subtracting 180° from it.

A bearing is always less than 360°, so it should be obvious whether adding or subtracting will give the correct reverse bearing.

Example
A boat leaves harbour at a bearing of 78°. After sailing for two hours at 20 km/h she makes an anticlockwise turn of 90° and sails for another two and a half hours at 17 km/h.

a) How far is the boat from harbour? Answer to the nearest 100 m.
b) To the nearest degree, what bearing should it sail to return to harbour?

First we must convert time and speed to distance:
First part of journey: 2 hours @ 20 km/h = 40 km
Second part of journey: 2.5 hours @ 17 km/h = 42.5 km.

Again, we draw a well-labelled diagram with an accurate scale. It is extremely important in problems like this.

a) From the diagram $BH^2 = HA^2 + AB^2$

$= 40^2 + 42.5^2$

$= 3406.25$

Hence the distance back to the harbour is

$\sqrt{3406.25} = 58.4$ km.
b) From the diagram, using alternate angles, if we can find the angle $\theta$ the required bearing would be $180^\circ + \theta$.

But $\theta$ can be found if we know $s\angle AHB$.

Now $s\angle AHB = \tan^{-1} \left( \frac{42.5}{40} \right) = 46.735704 \ldots^\circ$

$= 47^\circ$

Then $\theta = 90^\circ - (47^\circ + 12^\circ) = 31^\circ$

Hence the required bearing is $211^\circ$.

**Note:** With an accurate scale diagram, this problem could have been solved graphically using a ruler and compass.

**Activity 4.11**

Solve the following problems. Use 2 decimal places for both sides and angles.

1. A ship sails from a port in a direction S $27^\circ 21'$ W for 20 km. How far is she West of her starting point?

2. A ship bears $028^\circ 11'$ from a lighthouse and is a distance of 16 km from it. How far North of the lighthouse is the ship?

3. A canoeist begins at a landing area and paddles due south across a lake for 700 m. By changing direction to $270^\circ$ and paddling 430 m, the shore is reached. Calculate the:
   a) bearing
   b) distance required to paddle back to the landing area.

4. After sailing from a port, P, in a direction N $45^\circ 47'$ W a ship changes course to N $44^\circ 13'$ E. After a while it is found to be 20 km from P and bearing N $8^\circ 10'$ W. How far did it sail before changing course?

5. A ship is 5 km N $37^\circ$ W from a fort and a lighthouse is 12 km S $53^\circ$ W from the fort. Find the:
   a) bearing of the ship from the lighthouse
   b) bearing of the lighthouse from the ship.

**Check your answers at the end of this Section.**
4.12 Oblique triangles

An oblique triangle is one without a right angle. It could be an acute triangle, with all angles less than 90° or it could be an obtuse triangle in which one angle is more than 90°.

If we want to solve an oblique triangle we cannot use Pythagoras’ Theorem or the trigonometric ratios because these methods only apply to right triangles.

However, there are two rules that can be used. They are called the Sine Rule and Cosine Rule. What rule should be used depends on the information that is provided.

Obviously, every triangle has three sides and three angles ie six parts in total. Generally, knowing three of these parts will enable us to solve a triangle, including oblique ones. However, there are exceptions so it is best to consider each combination in turn.

1. Three sides

A triangle is uniquely determined when given three sides. This can easily be demonstrated with a pair of compasses. In geometry, this case is known as the congruency condition SSS for Side, Side, Side. Thus when given three side lengths of a triangle we should be able to find the three angle sizes.

2. Two sides and one angle

This case is not as straightforward because it depends how the angle is positioned in relation to the sides. In fact there are two possibilities:

• two sides and an included angle

Again, using a pair of compasses, it can be shown that a triangle is uniquely determined. The congruency condition is referred to as SAS for Side, Angle, Side.

• two sides and an angle not included

The SSA case is generally called the ambiguous case. The reason is that when given two sides and one angle not included, it is sometimes possible to draw two different triangles, one acute and one obtuse.

Consider this example.

Example 1  Draw a triangle with $A = 50^\circ$, $a = 6$ cm, $b = 8$ cm.

Observe the diagram.

The given information does not define a unique triangle.

In fact with the given information it is possible to draw two triangles, the obtuse triangle $AB_1C$ and the acute triangle $AB_2C$.
This verifies that when given any two sides and one angle, that is not included, we cannot always uniquely solve the triangle. In fact this case is **not** a congruency condition.

Looking at the diagram, notice that because \( \angle B_1 = \angle B_2 \), angle \( B_2 \) must be same size as the external angle \( B \) of the obtuse triangle \( AB_1C \).

Therefore, the obtuse angle \( B_1 \) in triangle \( AB_1C \) is the supplement of the acute angle \( B_2 \) in triangle \( B_1CB_2 \); the angles add to \( 180^\circ \).

This knowledge will help later in solving the two different triangles.

Sometimes on finding \( B_2 \) and consequently the obtuse angle \( B_1 \), the size of this obtuse angle \( B_1 \) added to the size of angle \( A \) will produce a total that exceeds \( 180^\circ \). In such a case only one possible triangle is defined. In fact using the configuration as in the diagram, it can be formally shown that two solutions are possible if:

- \( A \) is acute, and
- \( b \sin A < a < b \)

**Example 2**  Verify that if \( A = 60^\circ \), \( a = 5 \) and \( b = 6 \), only one solution is possible

Here \( b \sin A = 6 \sin 60^\circ = 5.1961... \)

This is not less than 5. Hence only one triangle can be drawn (and solved!) using the provided information.

3. **One side and two angles**

Knowing two angles automatically means that the third angle is known since the angles must add to \( 180^\circ \). When in addition a side length is given, a scale factor is introduced that uniquely determines the size and shape of the triangle. In fact this case is known as the congruency condition **AAS** for Angle, Angle, Side.

4. **Three angles**

It should be obvious that when given three angles, a triangle is not uniquely determined because different sized (similar!) triangles could be drawn, as is shown on the right.

For this reason, **AAA** could not be used to prove that two triangles are congruent i.e have the same shape and size. Indeed, **AAA** is not a congruency condition.
**Activity 4.12**

In each case, three features about a triangle ABC are given. Determine whether the triangle can be solved and if so, whether the solution is unique.

Remember: small letters indicate sides and capital letters indicate angles.

1. \( a = 7 \text{ cm} \) \( b = 5 \text{ cm} \) \( c = 9 \text{ cm} \)
2. \( A = 34^\circ \) \( a = 7 \text{ cm} \) \( b = 8 \text{ cm} \)
3. \( A = 34^\circ \) \( B = 78^\circ \) \( C = 68^\circ \)
4. \( A = 34^\circ \) \( B = 46^\circ \) \( c = 7.4 \text{ mm} \)
5. \( A = 128^\circ \) \( a = 6.3 \text{ m} \) \( c = 5.7 \)
6. \( A = 123^\circ \) \( b = 8 \text{ cm} \) \( c = 6 \text{ cm} \)

Check your answers at the end of this Section.

**4.13 The sine rule**

We use the sine rule when given the **AAS** case and the **SSA** case.

Consider the diagram on the right.

Here the perpendicular is dropped from A onto CB.

Now \( \sin C = \frac{AP}{b} \) i.e. \( AP = b \sin C \)

Also \( \sin B = \frac{AP}{c} \) i.e. \( AP = c \sin B \)

Thus \( b \sin C = c \sin B \) (both equal \( AP \))

Rearranging we have \( \frac{b}{\sin B} = \frac{c}{\sin C} \)

Now, if we drop a perpendicular from C to AB, as in the second diagram on the right, we have

\( b \sin A = a \sin b \) (both equal \( CQ \))

\( \frac{a}{\sin A} = \frac{b}{\sin B} \)

Combining the parts of the sine rule above, we have the **sine rule**.

\( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)
**Example 1** Find, correct to two decimal places, $b$ in triangle $ABC$ if $a = 43.7$, $B = 73^\circ 18'$ and $C = 44^\circ$.

This is an AAS case, thus there is one solution.

\[
A = 180^\circ - (B + C) = 180^\circ - (73^\circ 18' + 44^\circ) = 62^\circ 42'
\]

\[
\frac{b}{\sin B} = \frac{a}{\sin A} \quad \Rightarrow \quad b = \frac{a \sin B}{\sin A}
\]

i.e. \[
b = \frac{43.7 \sin 73^\circ 18'}{\sin 62^\circ 42'} = 47.10
\]

**Example 2** Correct to the nearest unit and nearest minute, solve triangle $ABC$, given $c = 314$, $b = 240$ and $C = 55^\circ 10'$.

This is the SSA case so there could be two solutions.

For $B$:

\[
\sin B = \frac{b \sin C}{c} = \frac{240 \sin 55^\circ 10'}{314} = 0.6274\ldots
\]

then $B = \sin^{-1}(0.6274) = 38^\circ 51'$

therefore $A = 180^\circ - (B + C) = 85^\circ 59'$

For $a$:

\[
a = \frac{b \sin A}{\sin B} = \frac{240 \sin 85^\circ 59'}{\sin 38^\circ 51'} = 382
\]

If we check for a possible second solution we note that $B$ may be $180^\circ - 38^\circ 51' = 141^\circ 10'$ which is impossible for the given triangle because $55^\circ 10' + 141^\circ 10' > 180^\circ$. 

---

*Note: The diagram is not included in the text.*
Example 3  Solve the triangle ABC, given \( a = 63 \), \( b = 103.6 \) and \( \angle A = 33 ^\circ 40' \).

For \( B \):

\[
\sin B = \frac{b \sin A}{a} = \frac{103.6 \sin 33^\circ 40'}{63} = 0.9116...
\]

\( B = 65^\circ 44' \)

\( C = 180^\circ - (A + B) = 80^\circ 36' \)

\[
c = \frac{a \sin C}{\sin A} = \frac{63 \sin 80^\circ 36'}{\sin 33^\circ 40'} = 112.1
\]

When \( \sin B = 0.9116 \), \( B \) can also equal \( 180^\circ - 65^\circ 44' = 114^\circ 16' \).

This situation is illustrated in the diagram below.

We have therefore, another set of solutions.

\( B = 114^\circ 16' \)

\( C = 180^\circ - (114^\circ 16' + 33^\circ 40') = 32^\circ 4' \)

\[
c = \frac{63 \sin 32^\circ 4'}{\sin 33^\circ 40'} = 60.3
\]
Activity 4.13
Solve the following triangles ABC given that

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>c = 25</td>
<td>A = 35°</td>
</tr>
<tr>
<td>2.</td>
<td>b = 321</td>
<td>A = 75.3°</td>
</tr>
<tr>
<td>3.</td>
<td>b = 215</td>
<td>c = 150</td>
</tr>
<tr>
<td>4.</td>
<td>C = 30°</td>
<td>b = 12</td>
</tr>
</tbody>
</table>

Check your answers at the end of this Section.

4.14 The cosine rule
We use the cosine rule when given the SSS case and the SAS case.

The rule is, referring to the figure below:

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ c^2 = a^2 + b^2 - 2ab \cos C \]

Rearranging these:

\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc} \]
\[ \cos B = \frac{a^2 + c^2 - b^2}{2ac} \]
\[ \cos C = \frac{a^2 + b^2 - c^2}{2ab} \]

Try to memorise \( a^2 = b^2 + c^2 - 2bc \cos A \) and practise constructing the others above from it. Note that this rule applies even when angle A, B or C is greater than 90°, in which case the cosine is negative and the term \(-2bc \cos A\) becomes positive.
The following derivation would help you appreciate the similarity between the cosine rule and Pythagoras’ Theorem. You may skip it if you want to as we are mainly concerned with the application of the rule.

\[ a^2 = h^2 + d^2 \]
\[ = b^2 - e^2 + d^2 \]
\[ = b^2 - e^2 + (c - e)^2 \] (c being the side opposite to C)
\[ = b^2 - e^2 + c^2 - 2ce + e^2 \]
\[ = b^2 + c^2 - 2ce \]

But \( e = b \cos A \)

Therefore, \( a^2 = b^2 + c^2 - 2bc \cos A \)

**Example 1** Find, correct to the nearest minute, the size of angle \( A \) in triangle ABC if \( a = 47; b = 30 \) and \( c = 25 \).

\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc} \]
\[ = \frac{(30)^2 + (25)^2 - (47)^2}{2(30)(25)} \]
\[ = \frac{900 + 625 - 2209}{(30)(50)} \]
\[ = \frac{-684}{1500} \]
\[ = -0.4560 \]

So \( A = 117.13^\circ \)
Example 2  
In triangle ABC, B = 60°, a = 4 cm, c = 7 cm. Solve the triangle.

As we are given two sides and the included angle we can therefore apply the cosine rule to find the third side.

\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ = 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 60° \]
\[ = 16 + 49 - 56 \times 0.5 \]
\[ = 37 \]
therefore \[ b = \sqrt{37} \]
\[ = 6.0828 \text{ cm} \]

To find angle A, the sine or cosine rule may be used, but the former is more useful.

\[ \sin A = \frac{a \sin B}{b} \]
\[ = \frac{4 \times \sin 60°}{6.0828} \]
\[ = \frac{4 \times 0.8660}{6.0828} \]
\[ = 0.5695 \]

therefore \[ A = 34.7° \]
and \[ C = 180° - A - B \]
\[ = 180° - 34.7° - 60° \]
\[ = 85.3° \]

Hence, a = 4 cm and A = 34.7°
b = 6.08 cm and B = 60°
c = 7 cm and C = 85.3°
Activity 4.14

Solve the following triangles ABC given that:

1. \( a = 120 \quad b = 270 \quad C = 118.7^\circ \)
   
   (Note for \( 90^\circ < x^\circ < 180^\circ \), \( \cos x^\circ \) is negative.)

   The cosine rule is used here initially to find \( c \),
   
   \[ c^2 = a^2 + b^2 - 2ab \cos C \]

   The sine rule should then be used to complete the solution of the triangle.

2. \( a = 525 \quad c = 421 \quad A = 130.8^\circ \)
3. \( b = 10 \quad c = 12 \quad B = 35^\circ \)
4. \( a = 8 \quad b = 10 \quad c = 9 \)

Check your answers at the end of this Section.

4.15 Practical oblique triangle problems

Note in the following examples that the first step to solving a problem is to describe it with a clear diagram.

Example 1

A man leaves a point walking at 6.0 km/h in a direction due west. Another man leaves the same point at the same time cycling at a constant speed 28.2 km/h in a direction S 37° W. Find how far the two men are apart after 4 hours.

In the diagram, \( AB (= c) \) and \( AC (= b) \) represent the distance travelled by the walker and the cyclist respectively. Their distance apart is \( BC (= a) \).

Since distance = (speed) (time)

\[ c = 6.0 \times 4 = 24 \text{ km} \]

\[ b = 28.2 \times 4 = 112.8 \text{ km} \]

The angle between \( AB \) and \( AC \)

\[ = 90^\circ - 37^\circ \]

\[ = 53^\circ \]

The required distance, \( BC \), can be found by applying the cosine rule (two sides and the included angle given) to triangle ABC:

\[ a^2 = c^2 + b^2 - 2bc \cos 53^\circ \]

\[ = 24^2 + 112.8^2 - 2(24)(112.8) \cos 53^\circ \]

\[ = 576 + 12723.84 - 3258.47 \]

\[ = 10041.4 \]

\( BC = 100.2 \text{ km} \)

After 4 hours the two men are 100.2 km apart.
Example 2  If in Example 1 the cyclist’s speed (constant) was unknown and the two men were 120 km apart after 4 hours, find the speed of the cyclist.

Again we can use the triangle ABC to describe the movements of the two men.

In triangle ABC, we have 2 sides and an angle (not the included one) as the given information:

\[ c = AB = 24 \text{ km} \]
\[ a = BC = 120 \text{ km} \]
\[ A = 53^\circ \]

Thus the sine rule should be used.

\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]

We are required to find \( b \), so that the cyclist’s speed can be calculated.

we use \[ \frac{b}{\sin B} = \frac{a}{\sin A} \]

or \[ \frac{b}{\sin B} = \frac{c}{\sin C} \]

But either of the above contains two unknown quantities \( b \) and \( \sin B \).

Therefore we need to find \( \sin B \) first as follows:

From \[ \frac{a}{\sin A} = \frac{c}{\sin C} \] we have

\[ \frac{120}{\sin 53^\circ} = \frac{24}{\sin C} \]
\[ \sin C = \frac{24 \sin 53^\circ}{120} \]
\[ = 0.1597 \]
\[ C = 9.2^\circ \]

(or 170.8, NOT possible because 170.8 + 53° > 180°)

Thus there is only one possible solution for triangle ABC.

\[ B = 180^\circ - (A + C) \]
\[ = 180^\circ - (53^\circ + 9.2^\circ) \]
\[ = 117.8^\circ \]

Hence,

\[ \frac{b}{\sin 117.8^\circ} = \frac{120}{\sin 53^\circ} \]
\[ b = \frac{120 \sin 117.8^\circ}{\sin 53^\circ} \]
\[ = 132.9 \]

The speed of the cyclist = \( \frac{\text{distance}}{\text{time}} \)
\[ = \frac{132.9}{4} \]
\[ = 33.2 \text{ km/h} \]
Example 3

Walking towards a tower at 8 km/h, a man observed at a particular time that the angle of elevation of its top was 10.2°, and 3 minutes afterwards that it was 32°. Find the tower’s height and distance from the second point of observation.

The **angle of elevation** of the tower top from an observer is the angle between the line of sight and the horizontal, as shown in the diagram.

In the diagram (NOT to scale):
A and B are the first and second points of observation respectively.
CD is the tower, whose height, \( h \) is required.

Before we can use the triangle ABC (or BCD), to find \( h \) and the required distance, \( BD \), we need to know \( b \) (or \( a \)).

We will use triangle ABC to find \( b \), with the following information:

\[
\begin{align*}
A &= 10.2° \\
\text{time} &= \frac{3}{60} \text{ hour} \\
c &= AB = \text{speed} \times \text{time} \\
&= 8 \times \frac{3}{60} \text{ km} \\
&= 0.4 \text{ km} \\
\angle ABC &= 180° - 32° = 148°
\end{align*}
\]

With two angles and a side given, the sine rule should be used.

Since \( c = 0.4 \), we want
\[
\angle ACB = 180° - (10.2° + 148°) = 21.8°
\]

The sine rule gives:

\[
\frac{c}{\sin \ 21.8°} = \frac{b}{\sin \ 148°}
\]

\[
b = \frac{0.4 \sin \ 148°}{\sin \ 21.8°}
\]

\[
= 0.5708
\]

Therefore

\[
h = b \sin 10.2°
\]

\[
= 0.57 \sin 10.2°
\]

\[
= 0.10
\]

\[
AD = b \cos 10.2°
\]

\[
= 0.56
\]

\[
BD = 0.56 - 0.4
\]

\[
= 0.16
\]

The height of the tower is 0.1 km (or 100 m) and it is 0.16 km from the second point of observation.
Example 4  A ship is observed from the top of a cliff, 152 m high, in a direction S 28° 19’ W at an angle of depression 8° 46’. Six minutes later the same ship is seen in a direction W 17° 13’ N at an angle of depression 9° 52’. Calculate the speed of the ship.

\[ \tan 8° 46' = \frac{152}{BX} \Rightarrow BX = \frac{152}{\tan 8° 46'} = 985.65 \text{ m} \]

\[ \tan 9° 52' = \frac{152}{BY} \Rightarrow BY = \frac{152}{\tan 9° 52'} = 873.92 \text{ m} \]

Using the cosine rule in \( \triangle XYB \):

\[ \angle B = 17° 13' + (90° - 28° 19') \]
\[ = 17° 13' + 61° 41' \]
\[ = 78° 54' \]

\[ XY^2 = 985.65^2 + 873.92^2 - 2(985.65)(873.92) \cos 78° 54' \]
\[ XY = 1184.72 \text{ m} \]

So the ship has sailed 1.185 km in 6 min
\[ = 11.85 \text{ km in 60 min} \]

So the average speed is 11.85 km/h
Example 5  
From a point $P$ at ground level, a mine shaft is constructed to a depth of $0.264$ km. Tunnels are constructed from $Q$, $3.450$ km due west of $P$ and from $R$ $2.875$ km and $S$ $40^\circ$ E of $P$, to meet at the base of the shaft, $S$. Assuming $P$, $Q$ and $R$ are on horizontal ground and the two tunnels descend uniformly, calculate their angles of descent and the angle between the directions of the tunnels.

\[
\begin{align*}
\text{In } \triangle QPS & \quad \tan Q = \frac{0.264}{3.450} = 0.0765 \\
& \quad \angle Q = 4.38^\circ \\
\text{In } \triangle PRS & \quad \tan R = \frac{0.264}{2.875} = 0.0918 \\
& \quad \angle R = 5.25^\circ \\
\text{In } \triangle PQR & \quad QR^2 = 3.450^2 + 2.875^2 - 2(3.450)(2.875) \cos 130^\circ \\
& \quad QR^2 = 32.92 \\
& \quad QR = 5.738 \text{ km} \\
QS^2 & = 3.450^2 + 0.264^2 \quad RS^2 = 2.875^2 + 0.264^2 \\
QS^2 & = 11.972 \quad RS^2 = 8.335 \\
QS & = 3.460 \text{ km} \quad RS = 2.887 \text{ km} \\
\end{align*}
\]

\[
\begin{align*}
\text{In } \triangle QRS & \quad \cos \angle S = \frac{3.46^2 + 2.887^2 - 5.738^2}{2(3.46)(2.887)} \\
& \quad \cos \angle S = \frac{12.618}{19.978} = 0.6316 \\
& \quad \angle S \approx 129.17^\circ
\end{align*}
\]
Activity 4.15

1. An architect designing a building wants the roof overhang $AB$ on the north side such that in mid-summer when the noon altitude of the sun is $80^\circ$, the $10 \text{ m}$ wall $AC$ is just in shadow.

   Find the length $AB$ to the nearest 0.01 m, given angle $CAB$ has size $70^\circ$.

   \[ \begin{align*}
   \angle CAB &= 70^\circ \\
   \angle ABC &= 10^\circ \\
   \angle ACB &= 80^\circ \\
   AB &= 10 \text{ m}
   \end{align*} \]

2. $A$ and $B$ are two points on opposite banks of a river. From $A$, a line $AC = 275 \text{ m}$ is laid off and the angles $CAB = 125^\circ 40´$ and $ACB = 48^\circ 50´$ are measured. Find the length of $AB$.

3. From $A$, a pilot flies $125 \text{ km}$ in the direction N $38^\circ \ 20´$ W and turns back. Through an error he then flies $125 \text{ km}$ in the direction S $51^\circ \ 40´$ E. How far and in what direction must he now fly to reach his intended destination $A$?

4. $A$ and $B$ are two points $650 \text{ m}$ apart on one bank of a straight river. $C$ is a point on the other bank, and angles $CAB, CBA$ are observed to be $46^\circ \ 23´$ and $67^\circ \ 38´$ respectively. Find the width of the river.

5. $AB$ is the base line of a survey, $C$ and $D$ two points on the same side of $AB$ which are visible from both $A$ and $B$. If $AB$ is $4 \text{ km}$ long and $DAB = 22^\circ \ 15´$, $DBA = 35^\circ \ 30´$. $CAB = 65^\circ$, $CBA = 78^\circ \ 30´$, find by calculation the distance $CD$ and angle $CDA$.

6. A man observes two towers each $10 \text{ m}$ high. The bearing of the first tower is N $15^\circ$ E and the angle of elevation of the top of the tower is $22^\circ \ 15´$. The bearing of the second tower is N $70^\circ$ E and the angle of elevation is $18^\circ \ 42´$. Find the distance between the two towers.

7. A coastguard situated at the top of a cliff $200 \text{ m}$ high observes a ship in a direction S $32^\circ$ W at an angle of depression of $9^\circ \ 14´$. Five minutes later the same ship is seen in a direction W $25^\circ$ N at an angle of depression of $10^\circ \ 51´$. Calculate the speed of the ship in km/h$^{-1}$.

*Check your answers at the end of this Section.*
Assessment 4

1. In the triangle ABC, right angled at B, $C = 25^\circ$ and $a = 18.5$ cm, calculate the length of the hypotenuse.

2. Calculate, correct to 4 decimal places, the value of:
   
a) $\sin^2 35^\circ + \cos 12^\circ - \sec 4.1^\circ$
   
b) $\tan^{-1} 1.65 + \cos^{-1} 0.4 - \sin^{-1} 0.15$

3. Explain, by using a diagram, how you could obtain the exact value of $\tan 30^\circ$.

4. Solve the following equation for $\theta$, correct to the nearest second given that is $\theta$ is acute: $2\cos \theta - 1 = 0$

5. Find, correct to two decimal places, the length of the side marked by $x$.

![Triangle with sides 42°, 15 cm, and unknown x]

6. An escalator inclined at an angle of $42.6^\circ$ moves at a constant speed of 0.6 m/s.
   
   A person steps on the escalator and reaches the top in 42 seconds. Through what vertical height has the person been lifted?

7. A horizontal force of 25.9 N and a vertical force of 34.8 N are applied simultaneously. Calculate:
   
a) the resultant force
   
b) the angle with the horizontal at which the resultant force is applied.

8. The pilot of a helicopter hovering 100 m above the bank of a river, observes the opposite bank to be at an angle of depression of $57^\circ$. How wide is the river?

9. A competitor in an orienteering exercise runs 4.2 km on a bearing of S $36^\circ$ E, then 3.8 km on a bearing of N $54^\circ$ E. Calculate:
   
a) how far he is from his starting point
   
b) the bearing to return to his starting position.

10. Explain why the SSA case is called the ambiguous case.

11. Solve triangle ABC with $A = 45^\circ$, $b = 6$ mm and $c = 7$ mm.

12. In the triangle ABC with $\angle ABC = 64^\circ$, $AB = 35$ mm and $CA = 48$ mm. Calculate the size of $\angle CAB$. 
13. In the triangle shown below, calculate the length $p$ of side $\overline{QR}$.

![Triangle Diagram]

14. A tunnel is to be dug through a mountain from point $A$ to point $C$. From a point $B$ both points $A$ and $C$ are visible. If the angle $ABC$ is measured as $57° 15'$ and if $B$ is 1036 m from $A$ and 1348 m from $C$ calculate the length of the proposed tunnel.

15. A vertical mast stands on horizontal ground. A surveyor standing due south of the mast measures the angle of elevation of the top as $37° 28'$. He walks in a direction N $79° 40'$ W to a point 72 m from the base of the mast which is now on a bearing of N $17° 50'$ E.

Calculate the height of the mast.

**Answers to activities**

**Activity 4.1**

1.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$\sin A$</th>
<th>$\cos A$</th>
<th>$\tan A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10°$</td>
<td>0.1736</td>
<td>0.9848</td>
<td>0.1763</td>
</tr>
<tr>
<td>$25°$</td>
<td>0.4226</td>
<td>0.9063</td>
<td>0.4663</td>
</tr>
<tr>
<td>$40°$</td>
<td>0.6428</td>
<td>0.7660</td>
<td>0.8391</td>
</tr>
<tr>
<td>$53.2°$</td>
<td>0.8007</td>
<td>0.5990</td>
<td>1.3367</td>
</tr>
<tr>
<td>$68° 15' 20'$</td>
<td>0.9288</td>
<td>0.3705</td>
<td>2.5072</td>
</tr>
</tbody>
</table>

2. a) 1.0897   b) 0.5349
   c) 0.9971   d) 1.0724
   e) 0.0568   f) 1
Activity 4.2

1. $AD = \sqrt{3}$

2. $\sin 60^\circ = \frac{\sqrt{3}}{2}$
   
   $\cos 60^\circ = \frac{1}{2}$

3. $\sin 30^\circ = \frac{1}{2}$
   
   $\cos 60^\circ = \frac{\sqrt{3}}{2}$

   $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$
   
   $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Activity 4.3

1. $\sin^2 A + \cos^2 A = \left(\frac{a}{b}\right)^2 + \left(\frac{c}{b}\right)^2$

   $= \frac{a^2 + c^2}{b^2}$

   $= \frac{b^2}{b^2} = 1$

2. $RQ = \sin Q = \frac{5}{13}$

   $\cos Q = \frac{12}{13}$

   $\tan R = \frac{12}{5}$

Activity 4.4

1. a) 4.1336
   
   b) 1.3874
   
   c) 0.2484

2. $\csc \theta \geq 1 \quad \sec \theta \geq 1 \quad \cot \theta \geq 0$

3. $\cos \theta$
4. \( \sin^2 A + \cos^2 A = 1 \)

Divide both sides by \( \sin^2 A \)

\[ \frac{\sin^2 A}{\sin^2 A} + \frac{\cos^2 A}{\sin^2 A} = \frac{1}{\sin^2 A} \]

\[ 1 + \left( \frac{\cos A}{\sin A} \right)^2 = \left( \frac{1}{\sin A} \right)^2 \]

\[ 1 + \cot^2 A = \csc^2 A \]

Activity 4.5

<table>
<thead>
<tr>
<th>B</th>
<th>( \sin B )</th>
<th>( \cos B )</th>
<th>( \tan B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>47°</td>
<td>0.731</td>
<td>0.68</td>
<td>1.0724</td>
</tr>
<tr>
<td>30°</td>
<td>0.495</td>
<td>0.87</td>
<td>0.5697</td>
</tr>
<tr>
<td>52°</td>
<td>0.785</td>
<td>0.62</td>
<td>1.2655</td>
</tr>
<tr>
<td>1°</td>
<td>0.015</td>
<td>1.00</td>
<td>0.0146</td>
</tr>
</tbody>
</table>

2. \( 20° 19' \)  3. \( 78° 23' 47'' \)
4. \( 26.57° \)  5. \( X = 49° \)
6. a) \( 67° \)  b) \( 37° \)

Activity 4.6

1. a) \( 48° \)  b) \( 33° \)
2. \( 67° 23' \)

Activity 4.7

1. \( r = 20 \text{ m} \)
   \( s \angle R = 43.6° \)
   \( s \angle P = 46.4° \)
2. \( x = 13.8 \text{ cm} \)
   \( z = 10.0 \text{ cm} \)
3. \( e = 2.8 \text{ cm} \)
   \( d = 2 \text{ cm} \)
4. \( a = 2.6 \)
   \( c = 1.5 \)
5. \( s \angle F = 45° \)
   \( s \angle C = 30° \)
Activity 4.8

1. Angle with horizontal: 34.82°
   Total Force: 28.02 N

2. \( R = 22.5 \, \Omega \)
   \( X_L = 38.97 \, \Omega \)

3. a) 8.24 m
    b) 8.77 m

4. 2525.50 m

5. 79.47 cm

6. 13.79

7. 16.06 cm by 17.84 cm

8. 31.25 cm

9. a) 1045.28 m
    b) 9945.22 m

10. 7.28°

11. 9.10 cm

12. 28.26°

Activity 4.9

1. 39.44 m

2. 293.99 m

3. 81°

4. 103.78 m

5. 753.55 m

6. 404.29 m

7. 295.37 m

8. 17.45 km/h

Activity 4.10

1. a) 
   b) 
   c) 
   d) 

   N
   E
   W
   S
2. 

a) W 22° N 

b) S 80° W 

c) S 73° E 

d) W 15° S 

3. 

a) 

b) 

c) 

d)
Activity 4.11
1. 9.19 km
2. 14.10 km
3. a) 31.56°  b) 821.52 m
4. 15.84 km
5. a) 30.38°  b) 210.38°

Activity 4.12
1. Yes, unique
2. Yes, not unique
3. No, cannot be solved
4. Yes, unique
5. Yes, unique
6. Yes, unique

Activity 4.13
1. \( C = 77° \) \( a = 14.72 \) \( b = 23.79 \)
2. \( C = 66.2° \) \( c = 471.8 \) \( a = 498.8 \)
3. \( C = 28.2° \) \( A = 109.1° \) \( a = 299.7 \)
4. \( B = 36.9° \) \( A = 113.1° \) \( a = 18.39 \)
   OR \( B = 143.1° \) \( A = 6.9° \) \( a = 2.39 \)

Activity 4.14
1. \( c = 344.1 \) \( A = 17.8° \) \( B = 43.5° \)
2. \( C = 37.4° \) \( B = 11.8° \) \( b = 142.1 \)
3. \( C = 43.5° \) \( A = 101.5° \) \( a = 17.08 \)
4. \( A = 49.5° \) \( B = 71.8° \) \( C = 58.8° \)

Activity 4.15
1. 1.74 m
2. 2160 m
3. 29.02 km south west
4. 477 m
5. 4.938 km, 115° 4°
6. 25.34 m
7. 18.16 km/h
Section 5 – Linear Functions

5.1 The Cartesian plane

Rene Descartes, whose Latin name was Cartesius and who lived in the 17th Century, was the first person to develop a method for representing a point in the plane by a pair of real numbers.

A modified version of his method is still used today.

The Cartesian or rectangular coordinate system is formed by two perpendicular lines, the x-axis and the y-axis. The point of intersection of these axes is called the origin, denoted by 0.

Any point in this plane can be described by two values:
• how far left or right of the origin the point is ie its x value
• how far above or below the origin it is ie its y value.

The two values are referred to as the coordinates of the point and are always listed as a pair of numbers in brackets, separated by a comma and always listed in the order as detailed above. Hence the coordinates are also referred to as an ordered pair, \((x, y)\). Because a point is generally designated by a capital letter, the general notation for point P is \(P(x, y)\).

Locating a point in the Cartesian plane is called graphing or plotting the point. When a set of points is plotted, the result is called the graph.
Example 1  Graph the point (2, 3).

The notation (2, 3) means that the point is 2 units to the right of the origin and 3 units above it.

Example 2  Plot the points A(−1, 2), B(3, −1) and C(0, 4).
Activity 5.1

1. Identify the points A, B and C on the following graph.

2. Plot the points P(2, 4), Q(−2, 0) and R(−2, 4).

Check your answers at the end of this Section.

5.2 Types of graphs

Sometimes a special relationship exists between the $x$ and $y$ values of a set of points. Consider for example the ordered pairs $(1, 2)$, $(2, 4)$. Notice that the $y$ value is twice the $x$ value. We could write this in the form of an equation $y = 2x$. Other points that would satisfy this equation would be $(-2, -4)$, $(-1.5, -3)$, $(-0.22, -0.44)$ and $(0, 0)$. These points are plotted on the graph on the next page.

When the points are plotted, you discover that the points lie on a line through the origin. Any gaps between the points we have plotted could be filled up with an ordered pair that satisfies the equation $y = 2x$. There are an infinite number of points that all satisfy this equation and, if we could plot them all, there would be no gaps at all. Thus the equation $y = 2x$ determines a particular line.

Because there are no powers (other than 1) involved in the equation, we call an equation such as $y = 2x$ a **linear** equation.
The relationship between the $x$ and $y$ values of the points can be a lot more complicated.

Consider for example the points $(-1, 0)$, $(0, 2)$ and $(1, 6)$. Using advanced techniques it can be shown that these points follow the rule $y = x^2 + 3x + 2$ which is a quadratic equation. The curve produced is called a parabola. We will study this equation and its graph in the next Section.
Another important type of equation is the reciprocal equation which graphs into a hyperbola. This particular parabola is called a minimum parabola because it has a minimum.

Another important curve is the hyperbola. It is defined by a reciprocal equation. A typical example would be $y = \frac{2}{x}$.

A hyperbola has two branches that are in opposite quadrants. In the hyperbola drawn above, as one value gets bigger, either in the positive or negative direction, the other gets smaller. In fact, the graph gets closer and closer to the axes without intersecting them. We say that the axes are asymptotes of the graph.

Each of the three types illustrated has distinctive equations and distinctive features and you should be able to identify each type. Later in this unit, we shall pay special attention to the line and learn how to graph the line using special features. We shall also take a closer look at the parabola and hyperbola, both of which belong to the conic section family of curves.

We finish this topic with a summary table and some examples.

<table>
<thead>
<tr>
<th>Name of Equation</th>
<th>Equation</th>
<th>Name of Graph</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>$y = mx + b$</td>
<td>Line</td>
<td>Single power of $x$</td>
</tr>
<tr>
<td>Quadratic</td>
<td>$y = ax^2 + bx + c$</td>
<td>Parabola</td>
<td>Power of $x$ is squared</td>
</tr>
<tr>
<td>Reciprocal</td>
<td>$y = \frac{k}{x}$</td>
<td>Hyperbola</td>
<td>$x$ is in the denominator</td>
</tr>
</tbody>
</table>

In the table, the equations are written in a standard form but sometimes that is not the case and you have to carry out some simple algebra to rewrite the equations in standard form so that they can be identified. Let’s study a couple of examples.
Section 5 Linear Functions

Example 1  What type of graph is represented by the equation $2x + y = 8$?

By transposing the term $2x$, this equation can be rewritten as $y = -2x + 8$ which means that a straight line would result on graphing.

Example 2  What type of equation is represented by $xy = -9$?

Cross-multiplying $x$ produces $y = \frac{-9}{x}$, which is a reciprocal equation.

Example 3  Is the equation $x^2 + y^2 + 2x = 9$ the equation of a parabola?

On first glance this appears to be the case but we note that the $y$ is also squared. Thus the equation is not that of a parabola. In fact, the given equation would graph into a circle.

Activity 5.2

Identify each of the following as either a line (L), a parabola (P), a hyperbola (H) or neither (N).

a) $2x + 4 = 8$   b) $2xy = 8$   c) $x^2 - y^2 = 8$

d) $2x^2 - 7x - 6 = y$   e) $y = \frac{2x}{3}$   f) $x = -y + 7$

Check your answers at the end of this Section.

5.3 Graphing a line

Drawing a line that is defined by a linear equation is the easiest of all graphs to draw. This is because a line is completely determined by two points. Hence we only need to find the coordinates of two points, plot those points and connect them up. To be sure that we have not made a mistake, it is worthwhile to use a third point as a check.

When we select the points we generally select ‘easy points’. Those are points where one of $x$ or $y$ is 0; they are the intercept (with the axes) points.

You will find it convenient to set up a coordinate table as shown.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
In summary
- Set up a table.
- Calculate \( y \) given \( x = 0 \) and calculate \( x \) given \( y = 0 \).
- Select another suitable \( x \) value and calculate the corresponding \( y \) value.
- Plot the points and draw a line through the points.

**Example 1**  Graph the line \( y = x + 1 \)

**Example 2**  Graph the line \( y = -2x - 1 \)
Activity 5.3
Using three points and a table, graph the lines with the following equations.

a) \( y = x - 1 \)  
b) \( y = 2x - 1 \)  
c) \( x + y = 3 \)

Check your answers at the end of this Section.

5.4 Using the gradient

You will have noticed that the lines in the examples of the previous topic have a different slope. In Example 1 the line slopes upwards or rises. It has what is called a positive gradient, positive because when \( x \) values increase, so do \( y \) values. The situation is reversed in Example 2. There the line slopes downwards or falls. Thus when \( x \) values increase, the \( y \) values decrease; the line has a negative gradient.

The symbol for gradient is \( m \). Because gradients are used in calculating the equations of trend lines, an important practical topic with which we shall deal later, it is important that you can calculate \( m \) values. Before we show how this is done let us give you the formal mathematical definition of gradient.

The gradient \( (m) \) of a line is defined as:

\[
m = \frac{\text{rise}}{\text{run}}
\]

or

\[
m = \frac{\text{difference in } y \text{ values}}{\text{difference in } x \text{ values}}
\]

or

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

Finding the gradient is a matter of selecting two points on the graph for which we have the coordinates.

Example 1 Calculate the gradient of the line shown.

Taking the points \((0, -1)\) and \((1, 1)\), we clearly see that the line has a rise of 2 for a run of 1. Hence the gradient is 2.
Example 2  What is the gradient of this line?

Taking the points (0, 2) and (4, 0), the line rises −2 units (ie falls 2 units) for a run of 4 units. Hence \( m = \frac{-2}{4} = -0.5 \) .

Example 3  Calculate the gradient of the line passing through (−5, 9) and (2, −7).

Here it is best to use the formula with \( x_1 = -5, \ y_1 = 9 \) and \( x_2 = 2 \) and \( y_2 = -7 \).

Then \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 9}{2 - (-5)} = \frac{-16}{7} = -2.29 \)

rounded to two decimal places.

Note: There are two special cases:

- a horizontal line which has zero gradient
- a vertical line which has a gradient that is not defined.
**Example 4**  
Graph the lines $y = 3$ and $x = -2$ on the same diagram.

In the example the line $y = 3$ is a horizontal line. Because the line does not rise or fall its gradient is 0. Notice that in the table for $y = 3$, no matter what value of $x$ is selected, the $y$ value is always 3. You may wish to check that, using the formula, you will obtain $m = 0$.

The line $x = -2$ is a vertical line with undefined gradient. The gradient is undefined because it is infinitely large. Imagine a line with gradient of, say, 50 000. This means that for every unit increase in the positive $x$ direction, the $y$ value increases by 50 000 units. This line must be nearly vertical but we can even make it ‘more vertical’. In fact, the gradient value increases without bound, the more parallel to the line it becomes. When it is actually vertical, the gradient value cannot be given. Hence it is undefined.

Notice that in the table for $x = -2$, no matter what $y$ value is chosen, the $x$ value is always $-2$. Using the formula in this example results in 0 in the denominator. You may recall that dividing by 0 is not possible, providing another reason why vertical lines have gradients that are not defined.

**Activity 5.4**

1. Find the gradient of the following lines A and B.
2. Determine the gradient of the line through (−1, 5) and (5, 8).

3. Draw the graph of the line that has a gradient of −3 and passes through the point (1, 2).

4. Taxi fares are usually made up of a fixed charge and a charge depending on the number of kilometres travelled. In a particular town it cost $20 to travel 10 km and $25 to travel 20 km.

By calculating the gradient of the line with km on the x axis and total cost on the y axis, determine the rate per km.

Check your answers at the end of this Section.

5.5 Using the gradient form of a line

There are various ways to represent the equation of a line but the most useful way is by means of the gradient form.

You will recall that a line is determined by two points. In the following diagram, two special points are chosen. The first point is a point on the y axis represented by (0, b). The value b is called the y intercept of the line. The second point is a general point which we call (x, y).

Now we can determine the gradient of the line: \( m = \frac{y - b}{x - 0} = \frac{y - b}{x} \)

By cross multiplication we obtain \( mx = y - b \), and by transposition we obtain the gradient form \( y = mx + b \)

Thus knowledge of the gradient and the y intercept enables us to write down the equation of the line immediately.

Example 1  Write down the equation of the line through (0, −5) and with gradient 4.

Here \( m = 4 \) and \( b = −5 \). Hence the equation of the line is \( y = 4x - 5 \).

Example 2  What is the gradient and y intercept of \( 2x + y = 6 \)?

Here we must first transpose the \( 2x \) term. We obtain \( y = −2x + 6 \).

Thus the gradient of this line is −2 and the y intercept is 6.

Note: When graph paper is not of sufficient detail, we have to make informed guesses about \( m \) and \( b \).
Example 3  What is the equation of this line?

The $y$ intercept appears to be 0.5 while the gradient must be close to $-2$.
Hence the equation is likely to be $y = -2x + 0.5$.

Activity 5.5
1. What is the equation of the line with gradient $-10$ and $y$ intercept 8?
2. Obtain the gradient and $y$ intercept of the line with equation $2x - y = 7$.
3. What are the gradients and $y$ intercepts of lines A and B?

Check your answers at the end of this Section.
5.6 Using two points to obtain the line equation

Because a line is determined by two points, it is obvious that we should be able to use any two points to determine the equation of the line even if one is not the y intercept point.

The procedure is as follows.

- Obtain the value of $m$ first.
- Use this value and the coordinates of any of the two points and substitute into $y = mx + b$ to evaluate $b$.
- Write the equation as $y = mx + b$ using your values of $m$ and $b$.

**Example** 
Obtain the equation of the line through $(2, 5)$ and $(-4, 17)$.

Here $m = \frac{17 - 5}{-4 - 2} = \frac{12}{-6} = -2$

Using $(2, 5)$, we obtain $5 = -2(2) + b$ or $5 = -4 + b$

Hence $b = 9$.

Thus the equation is $y = -2x + 9$.

**Activity 5.6**

1. What are the equations of the lines $AB$, $AC$ and $BC$?

2. Obtain the equation of the line through $(2, -9)$ and $(3, 8)$.

Check your answers at the end of this Section.
5.7 Simultaneous linear equations using graphs

In Section 2, we solved simultaneous linear equations algebraically. A graphical approach can also be used. The intersection point of the lines (the graphs of the two linear equations), will then correspond to the solution.

An example will illustrate how simultaneous linear equations can be solved graphically.

**Example** Solve \( y = 2x - 4 \) and \( y = -x + 5 \) simultaneously by graphical means.

![Graph of simultaneous linear equations](image)

The solution is \((3, 2)\).

**Activity 5.7**

Graphically solve the following pairs of simultaneous equations:

1. \( y = x - 1 \) and \( y = 2x + 3 \)
2. \( y = 2x + 3 \) and \( y = x + 1 \)
3. \( y = -x + 5 \) and \( y = 2x - 1 \)
4. \( x + y = 7 \) and \( x - y = 1 \) (first rearrange)

Check your answers at the end of this Section.
5.8 Practical problems involving lines

Two variables may be related in many different ways. They may be related exponentially (e.g., population size and time), inversely (e.g., number of painters and time to paint a room) or linearly (e.g., taxi fare and distance travelled).

In this unit we only consider linear relationships that, as has been shown, can be represented by the equation \( y = mx + b \).

Let us study a simple example first.

**Example 1**

A car salesman is paid a basic wage of $1400 a week and a commission of $200 per car sold.

Express the weekly wage of the salesman \( y \) in terms of the number of cars \( x \) sold and draw a graph of \( y \) against \( x \). If the salesman had an income of $2000 in one particular week, how many cars did he sell in that week?

If he does not sell any cars in a week, that is if \( x = 0 \), he still earns $1400. Thus the \( y \) intercept is \( b = 1400 \).

Further, for every extra car he sells, he receives $200. Thus for an increase of \( x \) of one unit, \( y \) increases 200. Hence the gradient is \( m = 200 \).

The appropriate equation then becomes \( y = 200x + 1400 \) and the graph is drawn below. Notice that the graph starts at \( x = 0 \); you obviously cannot sell a negative number of cars!

![Graph](image)

We can calculate how many cars he sells in the week he earns $2000 by solving \( 2000 = 200x + 1400 \). Alternatively, we can use the graph and move sideways from \( y = 2000 \) until we 'hit' the graph and find that the corresponding \( x \) value is 3. Thus he sold 3 cars that week.

In many problems, it is more useful to use letters other than \( x \) and \( y \). A graph does not always have to be drawn.
Example 2  An apple farmer has a certain number of trees. Unfortunately, a disease starts killing the trees and they die off at a constant rate. After 3 weeks he has 925 left, while after 10 weeks only 750 are still alive. By expressing the number of trees alive \((n)\) in terms of the number of weeks \((w)\), determine how many trees the farmer had originally.

There are two ordered pairs given \((3, 925)\) and \((10, 750)\). The gradient of the line through these points is:

\[
m = \frac{750 - 925}{10 - 3} = \frac{-175}{7} = -25
\]

This means that 25 trees die off per week.

Let there be originally \(b\) trees. Then, using \(m = -25\) we obtain \(n = -25w + b\).

Substituting a point, say \((10, 750)\), produces \(750 = -25(10) + b\) or \(750 = -250 + b\).

Solving gives \(b = 1000\). Thus the farmer had 1000 trees originally.

Activity 5.8

1. The cost of hiring a tool is made up of two parts. There is a fixed charge and a charge of $5 per hour of use. If it cost a total of $125 for 10 hours of use, determine the equation connecting cost \((c)\) and the number of hours \((t)\) the tool is used.

2. In a certain city, the taxi fare is made up of two parts, a fixed charge and a rate per kilometre. If a 10 km trip costs $32 and a 31 km trip costs $57.20:
   a) Express total cost \((c)\) in terms of km \((k)\) using \(m\) for gradient and \(b\) for vertical axis intercept.
   b) Draw a graph of total cost \((c)\) against number of km \((k)\).
   c) Use the graph to approximate how far, to the nearest km, you could travel for $40.
   d) Check your estimate by calculation.

Check your answers at the end of this Section.
5.9 Using the ‘line of best fit’

In many practical problems we don’t have exact data points but points that have been estimated. This is typically the case in experiments where we have to rely on the accuracy of our measurement instruments to obtain information. This type of data is called empirical data.

For example, a biologist may have experimental data about the amount of fertiliser used on peas and pea production. She may plot the data points and suspect that the two variables involved are linearly related because the plotted points lie nearly on a straight line. Of interest to her would be to find the relationship equation and use this line, called the ‘line of best fit’ in this context, to make predictions about how much fertiliser should be used to produce a certain crop of peas.

The method we shall use here is:

- Use two points that are not ‘outliers’.
- Proceed as before to find the gradient and intercept and hence the equation of the line of best fit.

Note: There are more accurate methods available. In fact there are methods that use all the points, not just two. For the moment, however, we shall use the two-point method.

Example 1 The biologist collected the following empirical data

<table>
<thead>
<tr>
<th>Amount of Fertiliser</th>
<th>No. of Peas produced</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>5 g</td>
<td>200</td>
</tr>
<tr>
<td>10 g</td>
<td>325</td>
</tr>
<tr>
<td>20 g</td>
<td>490</td>
</tr>
<tr>
<td>40 g</td>
<td>510</td>
</tr>
</tbody>
</table>

Find the equation of the line of best fit and predict the number of peas produced when 15 g of fertiliser is applied.

Plotting the data clearly shows that the last point is an outlier. That point should not be used.
Using (5, 200) and (20, 490), we obtain \( y = 19.3x + 103.3 \).

Plotting the line of best fit we obtain the following diagram.

Looking at a highly magnified section of this graph shows that 15 g of fertiliser is likely to produce 393 peas. This can be confirmed by calculation since: when \( x = 15 \), \( y = 19.3(15) + 103.3 = 392.8 \).

In the last example, it is clear that the amount of peas produced ‘tapers off’ when more fertiliser is applied. Indeed, we have to be careful that we do not **extrapolate** outside the given interval of data.

The line of best fit equation we have derived is really only valid for amounts of fertiliser in the interval from 0 g to 20 g and even near the 20 g level, there may be some doubt.
Activity 5.9
The resistance of a component was measured at different temperatures in degrees Celsius and the following readings were taken.

<table>
<thead>
<tr>
<th>Temperature (T)</th>
<th>60 °C</th>
<th>80 °C</th>
<th>100 °C</th>
<th>120 °C</th>
<th>130 °C</th>
<th>150 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance (R)</td>
<td>55.4</td>
<td>58.6</td>
<td>61.8</td>
<td>64.8</td>
<td>66.0</td>
<td>60.5</td>
</tr>
</tbody>
</table>

a) Graph $R$ on the vertical axis against $T$ on the horizontal axis.
b) As well as you are able, draw a line through the points. It will not fit all points exactly.
c) The line may be represented by the equation $R = mT + b$. Use your graph to determine the values of $m$ and $b$.
d) Use your graph to estimate the resistance when $T = 105$ °C.
e) Check your estimate by calculation.
f) Would it be feasible to estimate the resistance when the temperature is 200 °C?

Check your answers at the end of this Section.

5.10 Polar coordinates
We are familiar with representing points in space by Cartesian coordinates that rely on an $x$ and a $y$ coordinate to mark how far a point is from the axes in two perpendicular directions. Another way of representing points in space is by means of polar coordinates.

In the polar coordinate system the position of a point is determined by:

- the length of a ray (the radius vector) from a fixed point (the pole)
- the angle (the polar angle) the ray makes with a fixed line (the polar axis).

We use the letter $r$ for the radius vector while $\theta$ is used for the polar angle.

Thus in the polar coordinate system, the position of a point is determined by distance ($r$) and direction $\theta$.

By convention, the angle is always measured from the polar axis in an anticlockwise direction. Also, at least in elementary work, the value of $r$ is always non-negative.
Contrary to Cartesian coordinates, polar coordinates are not unique. For example the points \((4, 50°)\), \((4, 410°)\) and \((4, -310°)\) all represent the same point. This may seem to be a disadvantage but, working in certain applications, it is much easier to work in polar coordinates.

Polar coordinates and graphs drawn using polar coordinates are best drawn on polar graph paper.

This consists of concentric circles with lines drawn at specified angles.

These angles are usually expressed in radians. Frequently fractions of \(\pi\) are used such as \(\frac{\pi}{6}\) which is equivalent to 30° since \(\pi \text{ rad} = 180°\).

**Example 1**   
Graph the point \(A(2, \frac{\pi}{6})\), \(B(4, \frac{3\pi}{4})\) and \(C(3, \frac{5\pi}{3})\)
Using elementary trigonometry it is possible to convert between Cartesian and polar coordinates. If we let the pole coincide with the origin and the polar axis with the x axis then the following graph enables us to see how the systems relate.

<table>
<thead>
<tr>
<th>To change form</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Polar to Cartesian</strong></td>
<td><strong>Cartesian to Polar</strong></td>
</tr>
<tr>
<td>( \cos \theta = \frac{x}{r} ) ie ( x = r \cos \theta )</td>
<td>( r^2 = x^2 + y^2 ) ie ( r = \sqrt{x^2 + y^2} )</td>
</tr>
<tr>
<td>( \sin \theta = \frac{y}{r} ) ie ( y = r \sin \theta )</td>
<td>( \tan \theta = \frac{y}{x} ) ie ( \theta = \tan^{-1}\left(\frac{y}{x}\right) )</td>
</tr>
</tbody>
</table>

**Example 2** Change \((1, 2)\) to polar coordinates. Use four decimal place accuracy.

Using \( r = \sqrt{x^2 + y^2} \) we obtain \( r = \sqrt{1^2 + 2^2} = \sqrt{5} = 2.2361 \)

Also \( \theta = \tan^{-1}\left(\frac{y}{x}\right) \). Thus \( \theta = \tan^{-1}\left(\frac{2}{1}\right) = \tan^{-1}(2) = 1.1071 \)

Consequently \((1, 2)\) is \((2.2361, 1.1071)\) in polar coordinates.

**Example 3** Change \((3, \frac{\pi}{3})\) to Cartesian coordinates. Use four decimal place accuracy.

Using \( x = r \cos \theta \) we obtain \( x = 3 \cos \frac{\pi}{3} = 1.5000 \)

And \( y = r \sin \theta \) gives \( y = 3 \sin \frac{\pi}{3} = 2.5981 \)

Thus \((3, \frac{\pi}{3})\) is equivalent to \((1.5000, 2.5981)\) in Cartesian coordinates.
Example 4  
Change the equation \( r = \sin \theta \) to a Cartesian equation.

Substituting \( \sin \theta = \frac{y}{r} \) we obtain \( r = \frac{y}{r} \) or \( r^2 = y \)

Since \( r^2 = x^2 + y^2 \) we obtain \( x^2 + y^2 = y \) or \( x^2 + y^2 - y = 0 \)

We finish this section with some examples involving polar equations and polar graphs.

Example 5  
Graph \( r = 2.5 \)

This simple equation represents a circle with radius 2.5.

![Graph of r = 2.5](image)

Example 6  
Draw the polar equation \( r = 2\theta \).

By graphing ordered pairs of the form \((r, \theta)\) such as \((2, 1)\) and \((4, 2)\) etc the so-called spiral of Archimedes is produced.

The spiral is a curve that winds around a fixed point and gradually recedes from the centre.

A spider creates its web in this manner.

![Graph of r = 2\theta](image)
Activity 5.10
1. Change (6, 135°) to Cartesian coordinates.
2. Change (4, 2) to polar coordinates.
3. Change (7, $\pi/4$) to Cartesian coordinates.
4. Change the following polar equations to Cartesian equations
   a) $r = 7$
   b) $r = \cos \theta$
5. Change the equation $x = 3$ to polar form.
6. Make a sketch of the spiral $r = \frac{\theta}{2}$.

Check your answers at the end of this Section.

5.11 Distance and mid-point

Consider the diagram below. We are trying to establish a formula for the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Notice that triangle PQR is a right triangle. We wish to find $d$, the length of the hypotenuse. Now we know the two other sides that make up the triangle and thus we can use Pythagoras’ Theorem:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and hence

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
**Example 1**  Find the distance between (2, −1) and (3, 4).

Applying the formula, 
\[
d = \sqrt{(3 - 2)^2 + (4 - (-1))^2} \\
= \sqrt{1^2 + 5^2} = \sqrt{26} = 5.1 \text{ (1 dp)}
\]

**Example 2**  The distance between A(1, 0) and B(4, q) is 5 units. Find q and state the coordinates of B.

Using the squared expression for distance, we obtain:
\[
5^2 = (4 - 1)^2 + (q - 0)^2 \\
\text{or } 25 = 9 + q^2
\]

Hence \( q^2 = 16. \)

We would be inclined to conclude that \( q = 4 \) but there is another solution and that is \( q = -4 \) since both \((-4)^2\) and \((4)^2\) equal 16.

Thus the points are B(4, 4) or B(4, −4).

The mid-point of a line segment is the average of the corresponding coordinates of the end points of the segment. In formula form:

Mid-point of \( PQ \) = \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)

**Example 3**  Find the mid-point of (1, −1) and (3, 5).

Applying the mid-point formula:

\[
\text{Mid-point} = \left( \frac{1+3}{2}, \frac{-1+5}{2} \right) = (2, 2)
\]
Activity 5.11

1. Find the distance between the points whose coordinates are \((-3, 1)\) and \((3, -7)\).

2. Suppose that \(A(-4, 2), B(0, 0)\) and \(C(3, 0)\) are the vertices of a triangle. Is triangle ABC isosceles?

3. Show that the points \(A(7, 5), B(2, 3)\) and \(C(6, -7)\) are the vertices of a right triangle. Also calculate the area.

4. Find, correct to two decimal places the perimeter of the triangle whose vertices are \((-2, 5), (4, 3)\) and \((7, -2)\).

5. Determine the coordinates of the mid-point for \((-3, 6)\) and \((3.0)\).

6. Find the mid-point of the line joining \(A(3, -4)\) and \(B(-5, -20)\).

Check your answers at the end of this Section.
Assessment 5

1. Graph the points A(6, 0), B (5, 2), C(-1, 1) and D(0, 3).

2. Which of the points lie on the line $3x - y = 12$?
   A(4, 0), B(7, 4), C(6, 6), D(0, 12), E(-2, 5), F(-3, 3), G(5, 3), H(-1, -9), J(1, 4).

3. Write the equations of lines A and B.

4. Graph the line $y = 3 - 2x$.

5. Find the gradient of the line $5x + 3y = 1$.

6. What is the equation of the line with gradient 5 and passing through the point (0, -2)?

7. Write down the equation of the line passing through (5, -3) and (-2, 4).

8. By graphical means, find the simultaneous solution to $y = 3x + 2$ and $y = -x + 6$.

9. A table lamp manufacturer has a fixed weekly outlay independent of the number of lamps made. He sells 50 lamps in a certain week and incurs a loss of $200. In the following week he sells 60 lamps and makes a profit of $200.
   a) Express his weekly profit ($P$) as a linear equation involving the number of lamps sold ($n$).
   b) How much does he sell each lamp for?
   c) What is his weekly outlay?
   d) How many lamps should he sell to make a profit of $1000 per week?
   e) What is his break-even point? (The break-even point is the number of lamps he should sell to ensure that he does not make a loss.)
   f) Illustrate the information on a well-labelled diagram.

10. Convert the Cartesian coordinate point (-2, 5) to polar coordinates. Use 2 decimal places accuracy.

11. Convert $(2, \frac{\pi}{6})$ to Cartesian coordinates. Use 2 decimal places accuracy.

12. Draw the polar graph $r = 4$ for $0 \leq \theta \leq \pi$.

13. Find the mid-point between (2, -6) and (-4, 9).

14. Find, correct to 2 decimal places, the distance between the polar coordinates (0, $2^6$) and (1, $4^8$).

15. Using three significant figures, what is the distance between (-1, 4) and (4, -3)?
Answers to activities

Activity 5.1
1. A(5, −5), B(3, 0) and C(−1, 4)
2.

![Diagram of a coordinate plane with points](image1)

Activity 5.2
a) L
b) H
c) N
d) P
e) L
f) L

Activity 5.3
a)

![Diagram of a coordinate plane with line](image2)

b)

![Diagram of a coordinate plane with line](image3)

c)
Activity 5.4
1. a) Line A: \( m = 2 \)  
   b) Line B: \( m = -1 \)
2. \( m = 0.5 \)
3. 
   ![Graph of lines with slopes and intercepts marked]
4. 50 c per km

Activity 5.5
1. \( y = -10x + 8 \)
2. \( m = 2 \) and \( b = -7 \)
3. Line A: \( m = -3, b = 2 \)  
   Line B: \( m = 4, b = -1 \)

Activity 5.6
1. Line AB: \( y = \frac{2}{5}x + 3 \frac{1}{5} \)  
   Line AC: \( y = -2 \frac{1}{2}x - 5 \frac{1}{2} \)  
   Line BC: \( y = 2 \frac{1}{3}x - \frac{2}{3} \)  
2. \( y = 17x - 43 \)
Activity 5.7

1. 

2. 

3. 

4. 

Activity 5.8

1. \( c = 5t + 75 \)

2. a) \( c = mk + b \)
   \( \quad c = 1.2k + 20 \)

   c) Approximately 16.7 km

   d) \( 40 = 1.2k + 20 \)
   \[ 1.2k = 20 \]
   \[ k = \frac{20}{1.2} = 16.67 \]
Activity 5.9
1. a) and b)

![Graph of R = 0.15T + 46.31](image)

c) Points used:

\[ R = 0.15T + 46.31 \]

(Your answer may be different if you have used different points.)

d) 62
f) No too much outside the interval 90 – 150.

e) 62.06

Activity 5.10
1. (−4.2426, 4.2426)
2. (0.4721, 0.4636) correct to 4 d.p.
3. (4.9497, 4.9497)
4. a) \( x^2 + y^2 = 49 \)
b) \( x^2 + y^2 - x = 0 \)
5. \( r = \frac{3}{\cos \theta} \)

Activity 5.11
1. 10
2. No
3. \( AB = \sqrt{29} \), \( BC = \sqrt{116} \), \( AC = \sqrt{145} \)
4. 23.56 units
5. (0, 3)
6. (−1, −12)
Section 6 – Quadratic Functions

6.1 Simplifying polynomials

An algebraic expression is a combination of numbers both variable and constant using the operations of arithmetic: addition, subtraction, multiplication, division and extraction of roots.

An example of an algebraic expression is: \( \sqrt{(x-8)^3} \)

Note that an algebraic expression is NOT an equation. An equation must contain an equal sign to show the relationship between two algebraic expressions.

An example of an equation is: \( \sqrt{(x-8)^3} = \frac{5}{3 - x} \)

An algebraic expression of the form \( 3bx^2 + 5ax - 7 \) contains three terms, \( 3bx^2, 5ax, \) and \( -7 \). The first term consists of the factors \( 3, b, \) and \( x^2 \). In the second term \( 5a \) is called the coefficient of \( x \).

Algebraic expressions with exactly one term are called monomials. Those having exactly two terms are binomials and those with exactly three terms are trinomials. In general, algebraic expressions with more than one term are called multinomials.

For example, \( 12a^2x \) is a monomial, \( 2x - 5 \) is a binomial and \( 3x^2 - 2x + 1 \) is a trinomial.

A polynomial in \( x \) is an algebraic expression of the form: \( a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \)

Where \( n \) is a non–negative integer, and the coefficients \( a_n, a_{n-1}, \ldots, a_1, a_0 \) are constants with \( a_n \) not zero. The degree of the polynomial is \( n \).

Thus \( 3x^3 + 7x^2 - 5x - 6 \) is an example of a polynomial of degree 3.

We shall mainly be concerned with the manipulation of monomials, binomials and trinomials. Let us recall how algebraic expressions such as polynomials should be simplified.
Example 1  Simplify $4(x + 3) - 2(x - 1)$.

$$4(x + 3) - 2(x - 1) = (4)(x) + (4)(3) - (2)(x) + (-2)(-1)$$

$$= 4x + 12 - 2x + 2$$

$$= 2x + 14$$

Example 2  Simplify $x(x - 2y) + y(x + y)$.

$$x(x - 2y) + y(x + y) = x^2 - 2xy + yx + y^2$$

$$= x^2 - 2xy + xy + y^2$$

$$= x^2 - xy + y^2$$

(Change $yx$ to $xy$ so it can be combined with $-2xy$.)

Example 3  Simplify $5k(3m + k) - 4m(4k + 3m)$

$$5k(3m + k) - 4m(4k + 3m) = 15km + 5k^2 - 16mk - 12m^2$$

$$= 15mk + 5k^2 - 16km - 12m^2$$

$$= 5k^2 - mk - 12m^2$$

Note:  We usually rearrange the answer so that the first term does not have a negative sign.

Activity 6.1

Multiply and simplify:

1. $b(b + 4) - 5(b + 2)$

2. $p(p^2 - 7p) + 3(2p^2 + 4)$

3. $x^2(xy - 3y^2) - y^2(2x^2 - xy)$

4. $a(a + b - 3) - b(a - b + 2)$

Check your answers at the end of this Section.

6.2 Multiplying binomials

When two binomials are multiplied, the whole first binomial expression must be multiplied by each term in the second.

$$(a + b)(c + d) = (a + b)(c) + (a + b)(d)$$

$$= (c)(a + b) + (d)(a + b)$$

$$= ca + cb + da + db$$

$$= ac + bc + ad + bd$$
We can produce this result by multiplying each term in the second bracket by each term in the first bracket.

\[(a + b)(c + d) = ac + ad + bc + db\]

This is generally the way multiplication of binomials is carried out because it is quicker. It is sometimes called the **FOIL** (First, Outside, Inside Last) method.

**Example 1** Multiply out \((x + 2)(y + 4)\).

\[(x + 2)(y + 4) = (x + 2)(y) + (x + 2)(4)\]
\[= y(x + 2) + 4(x + 2)\]
\[= xy + 2y + 4x + 8\]

or

\[(x + 2)(y + 4) = xy + 4x + 2y + 8\]

**Note:** These results are the same if we rearrange the order of the terms. The important point is that the same terms appear in both expressions, even though not in the same order.

**Example 2** Express without brackets \((x - 2)(3y + 4)\).

\[(x - 2)(3y + 4) = (x - 2)(3y) + (x - 2)(4)\]
\[= (3y)(x - 2) + 4(x - 2)\]
\[= 3xy - 6y + 4x - 8\]

or

\[(x - 2)(3y + 4) = 3xy + 4x - 6y - 8\]

If both brackets contain the same letter or literal part, we end up with ‘like terms’ which should be combined.

This process is illustrated in the following examples.
**Example 3** Simplify \((m + 2)(m + 3)\)

\[(m + 2)(m + 3) = m(m + 3) + 2(m + 3)\]
\[= m^2 + 3m + 2m + 6\]

We can see that the two middle terms are ‘like terms’, so we can combine them.

\[\therefore (m + 2)(m + 3) = m^2 + 5m + 6\]

**Example 4** Simplify \((x + 1)(x - 2)\).

\[(x + 1)(x - 2) = x(x - 2) + 1(x - 2)\]
\[= x^2 - 2x + x - 2\]
\[= x^2 - x - 2\]

Again the \(-2x\) and \(x\) are ‘like terms’ and hence can be combined.

**Example 5** Multiply \((2x - 3)(x - 2)\).

\[(2x - 3)(x - 2) = 2x(x - 2) - 3(x - 2)\]
\[= 2x^2 - 4x - 3x + 6\]
\[= 2x^2 - 7x + 6\]

A special case of this procedure is considered in the last two examples.

You will remember that \(x^2 = x \cdot x\) so it follows that \((a + b)^2 = (a + b)(a + b)\).

**Example 6** Simplify \((2x + 4)^2\).

\[(2x + 4)^2 = (2x + 4)(2x + 4)\]
\[= 2x(2x + 4) + 4(2x + 4)\]
\[= 4x^2 + 8x + 8x + 16\]
\[= 4x^2 + 16x + 16\]

**Example 7** Find the square of \((2x - 5)\) using the FOIL method.

The square of \((2x - 5) = (2x - 5)^2\)

\[= (2x - 5)(2x - 5)\]
\[= 4x^2 - 10x - 10x + 25\]
\[= 4x^2 - 20x + 25\]
Activity 6.2

Multiply:
1. \((x + a)(y + b)\)  
2. \((2x - 1)(4y + 3)\)  
3. \((3p - q)(2p - 5q)\)  
4. \((x^2 + 6)(5x - 1)\)  
5. \((3 - xy)(4 + y)\)  
6. \((x^2 - yz)(ax - z^2)\)

Multiply and simplify:
7. \((y + 2)(y + 4)\)  
8. \((x - 6)(x + 5)\)  
9. \((m - 1)(m - 5)\)  
10. \((x + 7)(x - 3)\)  
11. \((2x - 5)(x - 4)\)  
12. \((5x + 2y)(x + 4y)\)  
13. \((8p - 3q)(7p + q)\)  
14. \((r^2 - 1)(r^2 + 2)\)

Express the following squares as trinomials:
15. \((x - 2)^2\)  
16. \((2m - 3)^2\)  
17. \((6 - y)^2\)  
18. \((2m - 3n)^2\)  
19. \((x - 2y)^2\)  
20. \((-m + 3n)^2\)

Check your answers at the end of this Section.

6.3 Factorisation involving a common factor

Factorisation is the process of writing an expression in terms of factors.

For example the number 6 can be factorised and written as \(2 \times 3\).

Polynomials may also be factorised. The process involves the inverse of the operation just discussed; instead of removing brackets, we now insert them.

Note: Not all polynomials can be factored just as not all numbers (eg the prime numbers) can be factored.

The easiest type of factorisation involves a common factor, a factor that occurs in two or more terms in a polynomial.

Consider: \(a(b + c) = (a)(b) + (a)(c) = ab + ac\)

In reversing this multiplication, we factorise, that is:
\(ab + ac = a(b + c)\)
Formally, the process is called **taking out** or **extracting** a common factor. Here a is the common factor. The following examples will illustrate this process in more detail.

**Example 1** Factorise $2x + 2y$

Compare $2x + 2y$

↑↑ ↑↑

with $ab + ac$

We can see that $a = 2$, $b = x$, $c = y$.

So, if $ab + ac = a(b + c)$ then $2x + 2y = 2(x + y)$

We can check by multiplying out: $2(x + y) = 2x + 2y$

**Example 2** Factorise $x^2 + 3x$.

$x^2 + 3x = (x)(x) + 3(x)$

$= x(x + 3)$

Check: $x(x + 3) = x^2 + 3x$

as required.

**Example 3** Factorise $pq – 3pr$.

$pq – 3pr = (p)(q) + (p)(-3r)$

$= p(q – 3r)$

Check: $p(q – 3r) = pq – 3pr$

as required.

**Example 4** Factorise $11x – 22y^2$.

$11x – 22y^2 = (11)(x) – (11)(2y^2)$

$= 11(x – 2y^2)$

Check: $11(x – 2y^2) = 11x – 22y^2$

as required.

To extend this procedure to allow for mixed factors, we need to find the **greatest common factor** (G.C.F.).

With numbers, the greatest common factor can be found by inspection. For example, the greatest common factor of 8, 24 and 36 is 4, since 4 is the largest number that divides evenly into all three numbers.

The greatest common factor of $x^3$, $x^5$ and $x^7$ is $x^3$ because $x^3$ is the highest power of $x$ that is common to all three. Note that the greatest common factor of three terms like this must always be the lowest power.

If you study the following examples carefully, the process will become clear.

**Example 5** Find the greatest common factor of $6x^7$, $4x^5$ and $10x^3$.

First consider the numbers (coefficients) in front of each term, 6, 4 and 10. The greatest common factor of these is 2, since 2 is the highest number which will divide evenly into each coefficient.
Secondly, consider the variable \( x \) in each term: \( x^7, x^5 \) and \( x^3 \). The greatest common factor of these is \( x^3 \) (remember that it will always be the lowest power).

So the greatest factor of \( 6x^7, 4x^5 \) and \( 10x^3 \) is \( 2x^3 \) (the product of the factors found above).

**Example 6**  
Factorise \( 12x - 8 \) by removing all common factors.

\[
12x - 8 = 4(3x - 2)
\]

G.C.F. Result of \( \frac{-8}{4} \)

Result of \( \frac{12x}{4} \)

**Example 7**  
Factorise \( 4x^2 - 6x^3 \) by removing all common factors.

\[
4x^2 - 6x^3 = 2x^2(2 - 3x)
\]

G.C.F. Result of \( \frac{-6x^3}{2x^2} \)

Result of \( \frac{4x^2}{2x^2} \)

**Example 8**  
Factorise \( 7x^2y - 14x^3y^3 - 21x^3y \)

\[
7x^2y - 14x^3y^3 - 21x^3y = 7x^3y(1 - 2y^2 - 3x)
\]

**Activity 6.3**

Factorise:

1. \( 4a + 8b \)
2. \( 2 - 6k \)
3. \( x^2 + xy \)
4. \( pq - 3q^2 \)
5. \( x^2y - x \)
6. \( xyz + z^3 \)
7. \( 3x + 6x^2 \)
8. \( 5x - 15x^2y \)
9. \( 8x^2 - 4x^3y \)
10. \( -15xy + 55xz \)
11. \( 2bc - 6b^2c + 12b^3 \)
12. \( x^6 - 4x^3 + x^5 \)
13. \( -8x^3 + 6x^2 - 14x^5 \)
14. \( 10x^2y^4 - 5x^2y^2 - 10xy^3 \)

Check your answers at the end of this Section.
6.4 Factorisation involving difference of squares

Previously we learned how to multiply two binomials. If these binomials represent the difference of two numbers \((x - k)\) and the sum \((x + k)\) of two numbers, we obtain:

\[
(x - k)(x + k) = (x - k)(x) + (x - k)(k)
\]
\[
= (x)(x - k) + (k)(x - k)
\]
\[
= (x)(x) + (x)(-k) + (k)(x) + (k) - k
\]
\[
= x^2 - xk + kx - k^2
\]
\[
= x^2 - k^2
\]

Hence this product equals the difference between the squares of the numbers ie \((x^2 - k^2)\). In reverse:

\[
x^2 - k^2 = (x - k)(x + k)
\]

or

\[
x^2 - k^2 = (x + k)(x - k)
\]

That is, take the square root of each term first, then the factors are the sum and difference of these square roots.

Example 1 Factorise \(x^2 - 16\).

Although \(x^2 - 16\) has no common factors, we can see that it is in the general form of a difference of 2 squares \((x^2 - k^2)\), where \(k^2 = 16\) ie \(k = 4\)

\[
\therefore \quad x^2 - 16 = (x - 4)(x + 4)
\]

Example 2 Factorise \(x^2 - 25\).

\[
x^2 - 25 = (x + 5)(x - 5)
\]

Example 3 Factorise \(x^2 - 9\).

\[
x^2 - 9 = (x + 3)(x - 3)
\]

Example 4 Factorise \(25 - (6x)^2\).

\[
25 - (6x)^2 = (5 + 6x)(5 - 6x)
\]

Example 5 Factorise \(49 - (2x)^2\).

\[
49 - (2x)^2 = (7 + 2x)(7 - 2x)
\]

If two processes of factorising are to be done, one of which is extracting a common factor, always deal with the common factor first.
In fact, often we can’t see the ‘difference of squares’ until the common factors are removed.

**Example 6**  Factorise $2x^2 - 18$.

$$2x^2 - 18 = 2(x^2 - 9) = 2(x^2 - (3)^2) = 2(x - 3)(x + 3)$$

**Example 7**  Factorise $5 - 20p^2$.

$$5 - 20p^2 = 5(1 - 4p^2) = 5((1)^2 - (2p)^2) = 5(1 - 2p)(1 + 2p)$$

**Example 8**  Factorise $81p^4 - 16$.

$$81p^4 - 16 = (9p^2 - 4)(9p^2 + 4) = (3p - 2)(3p + 2)(9p^2 + 4)$$

**Note:** In this example we factorised twice using ‘difference of squares’. Not many problems are as tricky as this last one but the example illustrates that you should always examine your answer and be satisfied that you cannot factorise any further.

**Activity 6.4**

Factorise:

1. $x^2 - 36$
2. $m^2 - 64$
3. $y^2 - 1$
4. $100 - x^2$
5. $p^2 - q^2$
6. $(2n)^2 - 49$
7. $1 - k^2$
8. $25 - (3t)^2$
9. $7 - 28y^2$
10. $4x^2 - 16$
11. $y - y^3$
12. $24x^2 - 150y^2$
13. $3x^2 - 75$
14. $27ab^2 - 3ac^2$

Check your answers at the end of this Section.
6.5 Factorisation of trinomials

In section 6.2, we multiplied binomials as follows:

\[(y + 7)(y + 3) = y^2 + 3y + 7y + 21 = y^2 + 10y + 21\]
\[(y - 7)(y - 3) = y^2 - 3y - 7y + 21 = y^2 - 10y + 21\]
\[(y + 7)(y - 3) = y^2 - 3y + 7y - 21 = y^2 + 4y - 21\]
and \[(y - 7)(y + 3) = y^2 + 3y - 7y - 21 = y^2 - 4y - 21\]

In this section we want to perform the reverse operation; that is, to factorise \(y^2 + 10y + 21\) as \((y + 7)(y + 3)\), ie the product of binomial factors.

To help us in this, we can arrive at some general patterns in factorising trinomials algebraic expressions of the general form \((y^2 + by + c)\) by studying the set of examples above.

- The first terms in each bracket are the factors of the first term in the trinomial \((y \cdot x = y^2)\).
- The last terms in each bracket are the factors of the last term in the trinomial \((3 \times 7 = 21)\).
- The middle term of the trinomial is the sum of the last terms in each bracket \((3 + 7 = 10, -3 - 7 = -10, -3 + 7 = 4, -7 + 3 = -4)\).

So, very roughly, we are looking for two numbers which when multiplied together give the end term of our trinomial and which, added together, give the middle term of our trinomial.

That is: \[x^2 + (A + B)x + (AB) = (x + A)(x + B)\].

Example 1 Factorise \(x^2 + 8x + 16\).

First, looking for numbers which multiply together to give 16 (the end term of our trinomial) we have as possibilities:

1, 16
2, 8
and 4, 4

However, of these, only 4, 4 add up to 8 (which is the coefficient of the middle term in our trinomial). Therefore, 4, 4 is the required pair.

So, \(x^2 + 8x + 16 = (x + 4)(x + 4)\).
Example 2  Factorise \( x^2 + 2x - 8 \).

We want two numbers which when multiplied give \(-8\) and which when added give \(2\) (the coefficient of the middle term). The possibilities are:

1, \(-8\)
8, \(-1\)
2, \(-4\)

and 4, \(-2\) which all multiply to give \(-8\).

But of these, only 4, \(-2\) add up to 2.

\[ x^2 + 2x - 8 = (x + 4)(x - 2) \]

Check : \[ (x + 4)(x - 2) = x(x - 2) + 4(x - 2) = x^2 - 2x + 4x - 8 = x^2 + 2x - 8 \]

The above method of trial and error may be simplified using the cross method.

Example 3  Factorise \( m^2 + 4m - 12 \).

We first set up a cross as follows:

Terms for first bracket

Terms for second bracket

Factors of the first term in our trinomial  Factors of the end term

Now the factors of the first term \((m^2)\) are \(m, m\) and the possible choice of factors \((A, B)\) for the end term are:

\(-1, \ 12\)
1, \(-12\)
2, \(-6\)
\(-2, \ 6\)
3, \(-4\)
\(-3, \ 4\)

We choose the pair which, after we cross multiply, add up to the middle term of our trinomial.

So, in our case, we want:

\((C \times B) + (D \times A) = 4m\)
So, our working will appear as:

\[ 1m \quad +6 \quad \text{Since } (m \cdot -2) + (m \cdot 6) \]
\[ = -2m + 6m \]
\[ = 4m \]
\[ 1m \quad -2 \]

\[ \therefore m^2 + 4m - 12 = (m + 6)(m - 2) \]

Check: \((m + 6)(m - 2)\)
\[ = m(m - 2) + 6(m - 2) \]
\[ = m^2 - 2m + 6m - 12 \]
\[ = m^2 + 4m - 12 \]

At first this method may not seem quite as simple, but it will handle all types of trinomials. As a further example:

**Example 4**

Factorise \(2x^2 - 7x + 3\)

First notice that this time the coefficient of the first term is 2, not 1. Now using the cross method.

\[ 2x \quad -1 \]
\[ 1x \quad -3 \]
\[ \uparrow \quad \uparrow \]

Factors of \(2x^2\) and factors of 3 are chosen such that:
\[ 2x \cdot (-3) + 1x \cdot (-1) = -6x - x \]
\[ = -7x, \text{ the middle term.} \]

\[ \therefore 2x^2 - 7x + 3 = (2x - 1)(x - 3) \]

Check: \((2x - 1)(x - 3)\)
\[ = 2x(x - 3) - 1(x - 3) \]
\[ = 2x^2 - 6x - x + 3 \]
\[ = 2x^2 - 7x + 3 \]
Example 5   Factorise $x^2 + 5xy - 24y^2$.

First notice that this time the end term involves a second variable. Take a special note of how this affects our solution.

\[
\begin{array}{c}
1x \\
+ 8y \\
1x \\
- 3y \\
factors of -24y^2 \\
\end{array}
\]

\[\therefore x^2 + 5xy - 24y^2 = (x + 8y)(x - 3y)\]

Check: 
\[\begin{align*}
(x + 8y)(x - 3y) &= x(x - 3y) + 8y(x - 3y) \\
&= x^2 - 3xy + 8xy - 24y^2 \\
&= x^2 + 5xy - 24y^2
\end{align*}\]

Example 6   Factorise $5x^2 - 26x + 5$.

\[
\begin{array}{c}
5x \\
- 1 \\
1x \\
- 5 \\
\end{array}
\]

\[\therefore 5x^2 - 26x + 5 = (5x - 1)(x - 5)\]

Check: 
\[\begin{align*}
(5x - 1)(x - 5) &= 5x(x - 5) - 1(x - 5) \\
&= 5x^2 - 25x - x + 5 \\
&= 5x^2 - 26x + 5
\end{align*}\]

Activity 6.5

Factorise:

1. $x^2 + 3x + 2$
2. $x^2 - x - 2$
3. $x^2 - 7x + 12$
4. $x^2 + 2x - 15$
5. $x^2 + 5x - 24$
6. $2x^2 + 9x + 7$
7. $3x^2 + 11x - 4$
8. $5x^2 - 13x - 6$

Check your answers at the end of this Section.
6.6 Algebraic fractions

In dealing with algebraic fractions, we frequently need to factorise. Let us first look at multiplication and division.

We cancel arithmetic fractions by factoring the numerator and denominator.

\[
\frac{12}{84} = \frac{(2)(2)(3)}{(2)(2)(3)(7)} = \frac{2 \times 2 \times 3 \times 1}{2 \times 2 \times 3 \times 7} = 1 \times 1 \times \frac{1}{7} = \frac{1}{7}
\]

‘Cancelling’ is effected by pairing up a factor in the numerator with an equal factor in the denominator and using the properties that \( \frac{a}{a} = 1 \), and \( k \times 1 = k \).

To simplify algebraic fractions we can proceed in exactly the same way.

**Example 1**

Simplify: \( \frac{x^2 - x - 6}{x^2 - 7x + 12} \)

First we completely factor the numerator and denominator.

\[
\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)}
\]

Pairing the common factors in the numerator and denominator and cancelling:

\[
\frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} = 1 \times \frac{x + 2}{x - 4}
\]

Using the property that \( k \times 1 = k \):

\[
1 \times \frac{x + 2}{x - 4} = \frac{x + 2}{x - 4}
\]

**Example 2**

Simplify: \( \frac{2x^2 - 3x + 1}{4x^2 - 1} \)

\[
\frac{2x^2 - 3x + 1}{4x^2 - 1} = \frac{(2x - 1)(x - 1)}{(2x - 1)(2x + 1)} = \frac{(x - 1)}{(2x + 1)}
\]

Because we cancel using factors, the simplification of products and quotients of rational algebraic expressions follows along similar lines.
Example 3  Simplify: \( \frac{(5t + 5) \times \frac{t^2 - 4t + 4}{t - 2}}{t^2 - 1} \)

Factor: \( \frac{(5t + 5) \times \frac{t^2 - 4t + 4}{t - 2}}{t^2 - 1} = \frac{5(t + 1) \times (t - 2)(t - 2)}{(t - 1)(t + 1)} \)

Pair like factors in the numerator and denominator:
\[
\frac{5(t + 1)(t - 2)(t - 2)}{(t - 1)(t - 2)(t - 1)} = \frac{5}{1} \times \frac{t + 1}{t + 1} \times \frac{t - 2}{t - 2} \times \frac{t - 2}{t - 1}
\]

Use \( \frac{a}{a} = 1 \) and \( k \times 1 = k \) to obtain:
\[
\frac{5 \times 1 \times 1 \times t - 2}{t - 1} = \frac{5(t - 2)}{t - 1}
\]

Quotients are found by using the definition of division: \( x \div \frac{a}{b} = x \times \frac{b}{a} \) and then proceeding as before.

Example 4  Simplify: \( \frac{p^2 + 3p}{p^2 + 2p - 3} \div \frac{p}{p + 1} \)

Use the definition of division:
\[
\frac{p^2 + 3p}{p^2 + 2p - 3} \div \frac{p}{p + 1} = \frac{p^2 + 3p}{p^2 + 2p - 3} \times \frac{p + 1}{p}
\]

Now proceed as before:
\[
\frac{p^2 + 3p}{p^2 + 2p - 3} \times \frac{p + 1}{p} = \frac{p(p + 3)}{(p + 3)(p - 1)} \times \frac{p + 1}{p}
\]
\[
= \frac{p \times p + 3 \times p + 1}{p \times p + 3 \times p - 1}
\]
\[
= 1 \times 1 \times \frac{p + 1}{p - 1}
\]
\[
= \frac{p + 1}{p - 1}
\]
Algebraic fractions are added using exactly the same algorithm as for arithmetic fractions:

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}
\]

We obtain the fractions to be added or subtracted in an equivalent form with common denominators, by using the property:

\[
\frac{p}{q} = \frac{p}{q} \times \frac{a}{a} = \frac{pa}{aq}
\]

When adding or subtracting algebraic fractions, it is best to choose the common denominator to be the product of all the denominators. Then it is easy to see which factor to multiply both numerator and denominator by.

Example 1  Simplify \( \frac{2}{x} + \frac{1}{x-1} \)

Choose the common denominator to be the product of the denominators: \( x(x - x) \) (There is no need to multiply this out.)

To obtain the first fraction with this as a denominator, we must multiply top and bottom by \( x - 1 \). The second fraction is obtained with the new denominator by multiplying top and bottom by \( x \). So:

\[
\frac{2}{x} + \frac{1}{x-1} = \frac{2}{x} \times \frac{x-1}{x-1} + \frac{1}{x} \times \frac{x}{x-1}
\]

\[
= \frac{2(x-1) + x}{x(x-1)}
\]

\[
= \frac{2x - 2 + x}{x(x-1)}
\]

\[
= \frac{3x - 2}{x(x-1)}
\]

Example 2  Simplify \( \frac{x - 1}{x - 2} - \frac{x + 1}{x + 2} \)

Obtain the fractions with a common denominator:

\[
\frac{(x - 1)(x + 2)}{(x - 2)(x + 2)} - \frac{(x + 1)(x - 2)}{(x + 2)(x - 2)}
\]
Now use the common denominator and multiply out the numerator to make it easier to subtract:

\[
\frac{x^2 + x - 2 - (x^2 - x - 2)}{(x - 2)(x + 2)}
\]

\[
= \frac{x^2 + x - 2 - x^2 + x + 2}{x^2 - 4}
\]

\[
= \frac{2x}{x^2 - 4}
\]

Notice that sometimes the denominator can be multiplied out to obtain a simpler final result.

In some cases, the denominators should be factorised first in order to determine the desired common denominator as in the example below.

**Example 3** Simplify \( \frac{1}{t^2 - 1} + \frac{t}{t^2 + 2t + 1} \)

\[
\frac{1}{t^2 - 1} + \frac{t}{t^2 + 2t + 1} = \frac{1}{(t - 1)(t + 1)} + \frac{t}{(t + 1)(t + 1)}
\]

\[
= \frac{t + 1}{(t - 1)(t + 1)} + \frac{t(t - 1)}{(t + 1)(t + 1)(t - 1)}
\]

\[
= \frac{t + 1 + t^2 - t}{(t - 1)(t + 1)^2}
\]

\[
= \frac{t^2 + 1}{(t - 1)(t + 1)^2}
\]

**Note:** In this last example the common denominator wasn’t the product of the two original denominators. It was easier to obtain the common denominator by working out the simplest string of factors that needed to be used in order that the denominators are the same and then multiplying top and bottom on each fraction so as to get that denominator. This happens when there are already some common factors in the denominators.
Activity 6.6

Simplify:

1. \( \frac{x^2 - 4}{x^2 - 2x} \)

2. \( \frac{3t^2 - 27t + 24}{2t^2 - 16t + 14t} \)

3. \( \frac{x^2 - y^2}{x + y} \cdot \frac{x^2 + 2xy + y^2}{y - x} \)

4. \( \frac{p^2 - 4}{p^2 + 2p} \times \frac{p^2}{p - 2} \)

5. \( \frac{2a - 2}{a^2 - 2a - 8} \div \frac{a^2 - 1}{a^2 + 5a + 4} \)

6. \( \frac{z^2 + 2z}{3z^2 - 18z + 24} \div \frac{z^2 - z - 6}{z^2 - 4z + 4} \)

7. \( \frac{1}{x - 1} + \frac{1}{x + 1} \)

8. \( \frac{2}{t} - \frac{1}{t + 1} \)

9. \( \frac{p}{p + 2} + \frac{1}{p + 1} \)

10. \( \frac{a + 1}{a - 1} + \frac{a - 1}{a + 1} \)

11. \( \frac{1}{x^2 - x - 2} + \frac{1}{x^2 - 1} \)

12. \( \frac{1}{t} - \frac{t}{t - 1} \)

Check your answers at the end of this Section.

6.7 Solving quadratic equations by factoring

A quadratic equation is an equation in which the unknown is squared.

The general format is \( ax^2 + bx + c = 0 \)

Because of the term \( x^2 \), quadratic equations generally have two solutions, but sometimes there is only one. It is also possible that there is no solution such as in the quadratic equation \( x^2 + 4 = 0 \) because here we require that \( x^2 = -4 \) which, is of course, impossible because no number squared—even a negative number—will give us a negative result.

There are various methods for solving quadratic equations. In this section we look at the factoring method. Because, as we observed earlier, not all polynomials can be factored, this method has some limitations.

Let us start with an example.
Example 1  Solve by factoring  \( x^2 + 5x + 4 = 0 \)

We can factorise the trinomial to get:

\[
x^2 + 5x + 4 = (x + 4)(x + 1)
\]

Now  \( x^2 + 5x + 4 = 0 \)

So  \( (x + 4)(x + 1) = 0 \)

If two quantities multiplied together give a result of zero, one or the other (or both) of the quantities must be zero.

Here we have the quantities \( (x + 4) \) and \( (x + 1) \) multiplying together to give zero.

So we can say either \( x + 4 = 0 \) or \( x + 1 = 0 \)

Now we have two first degree equations which we can solve by the methods of Section 2.

\[
x = -4 \text{ or } x = -1
\]

Check that these values do make the original statement true.

If \( x = -4 \),  \( x^2 + 5x + 4 = (-4)^2 + 5(-4) + 4 \)
\[
= 16 - 20 + 4
\]
\[
= 0
\]

If \( x = -1 \),  \( x^2 + 5x + 4 = (-1)^2 + 5(-1) + 4 \)
\[
= 1 - 5 + 4
\]
\[
= 0
\]

So these values of \( x \) do make the equation true.

The solutions are \( x = -4 \) or \( x = -1 \).

Example 2  Solve by factoring  \( x^2 - 8x + 12 = 0 \)

\[
x^2 - 8x + 12 = 0
\]

\[
(x - 6)(x - 2) = 0
\]

\[
x - 6 = 0 \quad \text{or} \quad x - 2 = 0
\]

\[
x = 6 \quad \text{or} \quad x = 2
\]

Example 3  Solve by factoring  \( x^2 + 5x - 6 = 0 \)

\[
x^2 + 5x - 6 = 0
\]

\[
(x + 6)(x - 1) = 0
\]

\[
x + 6 = 0 \quad \text{or} \quad x - 1 = 0
\]

\[
x = -6 \quad \text{or} \quad x = 1
\]

When the right hand side of the equation is not zero, it must be rearranged before factoring.
Example 4  Solve by factoring \( x^2 + x - 2 = 10 \)
\[
\begin{align*}
  x^2 + x - 2 &= 10 \\
  \text{Rearrange:} & \\
  x^2 + x - 12 &= 0 \\
  \text{Factor:} & \\
  (x + 4)(x - 3) &= 0 \\
  x + 4 &= 0 \quad \text{or} \quad x - 3 = 0 \\
  x &= -4 \quad \text{or} \quad x = 3
\end{align*}
\]

Example 5  Solve by factoring \( (y + 3)^2 = 16 \).
\[
\begin{align*}
  (y + 3)^2 &= 16 \\
  \text{Multiply out:} & \\
  y^2 + 6y + 9 &= 16 \\
  \text{Rearrange:} & \\
  y^2 + 6y - 7 &= 0 \\
  \text{Factor:} & \\
  (y + 7)(y - 1) &= 0 \\
  y + 7 &= 0 \quad \text{or} \quad y - 1 = 0 \\
  y &= -7 \quad \text{or} \quad y = 1
\end{align*}
\]

Example 6  Solve by factoring \( x^2 = 5x \).
\[
\begin{align*}
  x^2 &= 5x \\
  \text{Rearrange:} & \\
  x^2 - 5x &= 0 \\
  \text{This is not a trinomial, but we can extract a factor:} & \\
  x(x - 5) &= 0 \\
  \text{Again, we have two quantities multiplying together to give zero. So we can say:} & \\
  x &= 0 \quad \text{or} \quad x - 5 = 0 \\
  x &= 0 \quad \text{or} \quad x = 5
\end{align*}
\]
Example 7  Solve by factoring \((y + 2)(2y - 3) = 6(y - 1)\).

This is not in the form we require, so we multiply the terms out and collect them.

\[(y + 2)(2y - 3) = 6(y - 1)\]

Multiply out:
\[2y^2 - 3y + 4y - 6 = 6y - 6\]

Collect terms:
\[2y^2 + y - 6 - 6y + 6 = 0\]

Factor:
\[2y^2 - 5y = 0\]
\[y(2y - 5) = 0\]

\[y = 0\] or \[2y - 5 = 0\]
\[y = 0\] or \[2y = 5\]
\[y = 0\] or \[y = \frac{5}{2}\]

Example 8  Solve by factoring \(x^2 = 9\).

Again we need to rearrange to give zero on the right hand side.

\[x^2 = 9\] gives \(x^2 - 9 = 0\)

This is a difference of squares, so again we can use factorisation:

\[x^2 - 9 = 0\]
\[(x - 3)(x + 3) = 0\]

\[x - 3 = 0\] or \[x + 3 = 0\]
\[x = 3\] or \[x = -3\]

This method avoids the trap of looking at the problem and deciding by inspection that the solution is \(x = 3\). You can see that \(x = -3\) is also a solution.

Example 9  Solve by factoring \(\frac{x^2}{2} = 8\)

Multiply both sides by 2:
\[x^2 = 16\]
\[x^2 - 16 = 0\]
\[(x - 4)(x + 4) = 0\]

\[x - 4 = 0\] or \[x + 4 = 0\]
\[x = 4\] or \[x = -4\]
Activity 6.7

Solve by factoring:

1. \(x^2 - 2x + 1 = 0\)
2. \(x^2 + 2x - 15 = 0\)
3. \(x^2 - 14x = 0\)
4. \(x^2 = 3x + 28\)
5. \(p^2 = 100\)
6. \((k - 3)^2 = 4\)
7. \(x^2 + 3x + 2 = 12\)
8. \(5c^2 = 12c\)
9. \(7x^2 = 28\)
10. \(9y^2 = 1\)
11. \(x^2 - 7x = 0\)
12. \((v + 4)(v + 2) = 8(1 + 3v)\)

Check your answers at the end of this Section.

6.8 Solving quadratic equations by formula

Another method for solving quadratic equations is by a process called completing the square. We shall not study this process but mention it because if this process is employed on the quadratic equation in its most general form:

\[ax^2 + bx + c = 0\]

then we obtain the following general solutions to this equation:

\[x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\]

and

\[x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\]

These two solutions are written together using the \(\pm\) sign. The result is called the quadratic formula for finding the solution to quadratic equations.

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

Particular quadratic equations can be solved by substituting values for \(a, b, c\).

Example 1 Use the formula to solve \(3p^2 - 4p - 2 = 0\). Give your answers correct to two decimal places.

First determine the values for \(a, b, c\). \(a = 3,\ b = -4,\ c = -2\)

Now substitute into the formula. In this example the variable name is \(p\) so:

\[p = \frac{(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}\]

Hence

\[p = \frac{4 \pm \sqrt{16 + 24}}{6} = \frac{4 \pm \sqrt{40}}{6}\]

Then

\[p = \frac{4 + \sqrt{40}}{6}\] or \[p = \frac{4 - \sqrt{40}}{6}\]

\[p = 1.72\] or \[p = -0.39\]
Activity 6.8
Use the formula to solve these quadratic equations. Express the results correct to two decimal places.
1. \(2x^2 - 5x - 12 = 0\) 4. \(5p + 2 = 12p^2\)
2. \(x^2 - 3x = -2\) 5. \(b^2 + 3b + 1 = 0\)
3. \(4t^2 - t - 1 = 0\) 6. \(k^2 - 14k + 12 = 0\)

Check your answers at the end of this Section.

6.9 Quadratic equation word problems

As with all word problems you should carefully define the variable you wish to solve for by a statement such as 'let \(x\) be …' Then write down any other quantities in terms of \(x\), and look for a statement in the question that relates all the quantities.

Drawing a diagram is generally very useful.

Always give your answer in expanded written form, using correct units.

Example 1  The perimeter of a rectangular field is 500 m, and its area is 14 400 m\(^2\).

Find the length of the sides.

Let the length of one of the sides be \(\ell\) metres.

The length of the other side is \(250 - \ell\) metres, as shown in the following diagram.

\[
\begin{array}{c}
250 - \ell \\
\ell \\
\ell \\
250 - \ell \\
\end{array}
\]

Half perimeter = 250 m

Area = \(\ell(250 - \ell)\)

Now translate ‘its area is 14 400’ into an equation:

\[\ell(250 - \ell) = 14 400\]

ie \(250\ell - \ell^2 = 14 400\)

Transposing, we have:

\[\ell^2 - 250\ell + 14 400 = 0\]  (Do not put the unit ‘metres’ in the equation.)

\((\ell - 90)(\ell - 160) = 0,\) whence

\(\ell = 90\) m or \(160\) m

The other side is \(250 - 90 = 160\) m or \(250 - 160 = 90\) m

Therefore there is only one set of lengths in the final solution.

The sides are 90 m and 160 m.
Example 2  
A man travelled 108 km from A to B. He then travelled from B to A in \(4 \frac{1}{2}\) hours less at a speed of 2 km/h faster. At what speed did he travel from A to B?

Let the required speed be \(v\) (km/h).

Then the time required to travel from A to B is \(\frac{108}{v}\) hours.

From B to A, the speed is \(v + 2\) (km/h).

The time required \(= \frac{108}{v + 2}\)

Now, we can translate \(\frac{4 \frac{1}{2}}{2}\) hours less’ to an equation. Note: \(\frac{4 \frac{1}{2}}{2} = \frac{9}{2}\)

Multiplying each term by \(2v(v + 2)\) we have:

\[
108 \times 2v = 108 \times 2(v + 2) - 9v(v + 2)
\]

ie \[0 = -216v + 216v + 432 - 9v^2 - 18v\]

ie \[9v^2 + 18v - 432 = 0\]

\[\therefore 9: \quad v^2 + 2v - 48 = 0\]

\[(v - 6)(v + 8) = 0, \text{ whence}\]

\[v = 6 \quad \text{or} \quad v = -8 \text{ (discard)}\]

Final solution: The required speed is 6 km/h.

Activity 6.9
1. If a train travelled 5 km an hour faster it would take one hour less to travel 210 km. What time does it take?
2. The perimeter of one square exceeds that of another by 100 m; the area of the larger square exceeds three times the area of the smaller by 325 m\(^2\). Find the length of their sides.
3. A lawn 50 m long and 34 m wide has a path of uniform width round it. If the area of the path is 540 m\(^2\), find its width.
4. A hall can be paved with 200 square tiles of a certain size. If each tile were 1 cm longer each way it would take 128 tiles. Find the length of each tile.
5. Two rectangles contain the same area, 480 m\(^2\). The difference of their lengths is 10 m, and of their breadths 4 m. Find the dimensions of each rectangle.
6. A and B are two stations 300 km apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B in nine hours; the train from B reaches A four hours after they meet: find the rate at which each train travels.

Check your answers at the end of this Section.
6.10 The parabola

The **parabola** is the result of drawing a curve defined by the quadratic function

\[ y = ax^2 + bx + c. \]

The parabola is one of the more interesting and useful shapes that exist in our everyday life. When a ball is thrown it follows a parabolic path. Torches and telescopes use parabolic mirrors which ensure that light is reflected as a parallel beam. In order to ‘transport’ sound over long distances in a straight line, telephone signals are sent backward and forward using a parabolic dish. Even the Sydney Harbour Bridge is in the shape of a parabola as are all suspension bridges.

The coefficients \( a, b \) and \( c \) determine the shape and positioning of the parabola.

The coefficient \( a \) determines the steepness of the curve. Note that \( a \) cannot be 0, otherwise a line will result. In fact, if:

- \( a < 0 \), the parabola slopes down from a maximum point, and when
- \( a > 0 \), the parabola slopes up from a minimum point.

The coefficient \( b \) can be shown to largely determine horizontal positioning while the coefficient \( c \) determines vertical positioning.

Some sketches will clarify this information.

**A:** \( y = x^2 (a = 1) \) against

**B:** \( y = -x^2 (a = -1) \)

**Note:** The parabola has inverted.

**A:** \( y = x^2 (a = 1) \) against

**B:** \( y = 2x^2 (a = 2) \)

**Note:** The parabola has become steeper.
Section 6 Quadratic Functions

A: \( y = x^2 \ (b = 0) \) against

B: \( y = x^2 + x \ (b = 1) \)

Note: The parabola has shifted to the left (and downwards).

A: \( y = x^2 \ (c = 0) \) against

B: \( y = x^2 + 1 \ (c = 1) \)

Note: The parabola has shifted upwards.

All parabolas have a line of symmetry. It is a vertical line of the form \( x = k \), where \( k \) is a constant.

It can be shown that knowing the coefficients \( a \) and \( b \) completely determines the equation of the line.

In fact its equation is \( x = -\frac{b}{2a} \).

After we obtain this value, we can substitute into the equation of the parabola to find the minimum or maximum value. This point is generally called the turning point.

**Example 1** Find the minimum turning point of \( y = x^2 + 3x + 2 \).

Here \( a = 1 \), \( b = 3 \) and \( c = 2 \).

The line of symmetry has equation \( x = -\frac{b}{2a} = -\frac{3}{2 \times 1} = -1.5 \).

Substituting we obtain \( y = (-1.5)^2 + 3(-1.5) + 2 = -0.25 \)

Hence the minimum turning point is \((-1.5, -0.25)\).
To draw a parabola, it is generally sufficient to obtain the coordinates of the turning point and three or so other points on one side of the line of symmetry and then make use of the symmetric properties of the parabola.

Of course the coordinates of one point are always known and those are the coordinates of the y intercept of the parabola i.e. $(0, c)$.

**Example 2**  Draw the parabola $y = x^2 + 3x + 2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.5$</td>
<td>$-0.25$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$2$</td>
</tr>
<tr>
<td>$1$</td>
<td>$6$</td>
</tr>
</tbody>
</table>

**Example 3**  Draw the parabola $y = 3 - 2x - x^2$.

In order to determine $a$, $b$ and $c$ we have to rewrite the equation in the usual format i.e. $y = ax^2 + bx + c$.

Interchanging terms on the right produces $y = -x^2 - 2x + 3$.

Hence $a = -1$, $b = -2$ and $c = 3$.

This parabola will have a maximum point since $a < 0$.

The line of symmetry is given by $x = \frac{-b}{2a} = \frac{-(-2)}{2(-1)} = -1$.

If $x = -1$, $y = 4$. Hence the maximum turning point is at $(-1, 4)$.
Activity 6.10

Graph the following parabolas.

1. \( y = x^2 + 1 \)
2. \( y = 3 - 2x^2 \)
3. \( y = 2x^2 + x - 2 \)
4. \( y = 6 - x - x^2 \)
5. \( y = (x - 2)^2 \)

Check your answers at the end of this Section.

6.11 Simultaneous equations involving parabolas

In an earlier section we solved simultaneously two linear equations. Here we have a look at how a linear (graphing into a line) and a quadratic equation (graphing into a parabola) can be solved simultaneously.

If we graph the equations simultaneously, it should be clear that three cases can be distinguished.

<table>
<thead>
<tr>
<th>Line and parabola do not intersect.</th>
<th>Line and parabola intersect at one point (they touch).</th>
<th>Line and parabola intersect at two points.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
<td><img src="image3.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

To find the simultaneous solutions is equivalent to finding the intersection points. We can use algebra or draw an accurate graph.

The process involved is illustrated in the following example, in which we obtain two distinct solutions.

**Example**

Solve \( y = 2x^2 + x - 3 \) and \( y = 2x - 2 \) simultaneously using

a) algebra  

b) graphs

a) Algebraically we equate the two equations

\[ 2x^2 + x - 3 = 2x - 2 \quad \text{or} \quad 2x^2 - x - 1 = 0 \]

Factorising: \((2x + 1)(x - 1) = 0\)
And hence $x = -0.5$ or $x = 1$

The $y$ values can be obtained by substituting the obtained $x$ values in one of the equations (the linear one would be easiest!).

If $x = -0.5$, $y = -3$ and if $x = 1$, $y = 0$.

b) Drawing the graphs:

The intersections (and therefore the solutions!) are: $(-0.5, -3)$ and $(1, 0)$.

Activity 6.11

1. Consider $y = x^2 - 2x - 1$ and $y = -2x - 2$. By drawing sketches on the same diagram, determine which is correct.
   a) The graphs touch at one point.   b) The graphs intersect at two points.
   c) The graphs do not intersect.   d) The graphs intersect at one point.

2. Consider $y = -2x^2 - 4x - 1$ and $y = 2x + 2$. By drawing sketches on the same diagram, determine which is correct.
   a) The graphs touch at one point.   b) The graphs intersect at two points.
   c) The graphs do not intersect.   d) The graphs intersect at one point.

3. By algebraic means, solve simultaneously $y = x - 4$ and $y = -x^2 - 6x + 4$.

4. Draw, on the same diagram, accurate graphs of $y = x^2 - 2x - 1$ and $y = -x + 1$. Use your graphs to find the simultaneous solutions.

Check your answers at the end of this Section.
Assessment 6

1. Simplify $2x(1 - x) - 3(2 - x^2)$
2. Multiply out: $(2x - 3)(x + 5)$
3. Factorise $3t^2 - 6t$
4. Factorise $25 - 16p^2$
5. Factorise $x^2 - 6x + 9$
6. Factorise $4z^2 + 4z - 15$
7. Use the quadratic formula to solve the quadratic equation: $3t^2 - 2t - 2 = 0$
   Express the solutions correct to three decimal places.
8. Simplify $\frac{x^2 - 4}{x^2 + 2x - 3} ÷ \frac{x^2 - x - 6}{x^2 - 9}$
9. Simplify $\frac{1 - x}{x} ÷ \frac{1 - x}{1 - x}$
10. A rectangular plot 4 m by 8 m is to be used for a garden. It is decided to put a pavement inside the entire border so that 12 m$^2$ of the plot is left for flowers. How wide should the pavement be?
11. A rectangular garden is 2 m longer than it is wide. If the width is doubled and the length diminished by 4 m, the area is unchanged. What are the original dimensions?
12. Graph the parabola $y = 2x^2 - x - 1$ over the interval (domain) $-2 \leq x \leq 2$.
13. Graphically solve $-3x^2 - x + 3 = 0$.
   Hint: Draw the parabola $y = -3x^2 - x + 3$ and determine where the parabola crosses the x axis (i.e., where $y = 0$).
14. Simultaneously solve $y = x^2 - 2x - 3$ and $y = x + 1$. 
Answers to activities

Activity 6.1
1. \( b^2 - b - 10 \) 
2. \( p^3 - p^2 + 12 \) 
3. \( x^3y - 5x^2y^2 + xy^3 \) 
4. \( a^2 + b^2 - 3a - 2b \)

Activity 6.2
1. \( xy + xb + ay + ab \) 
2. \( 8xy + 6x - 4y - 3 \) 
3. \( 6p^2 - 17pq + 5q^2 \) 
4. \( 5x^3 - x^2 + 30x - 6 \) 
5. \( 12 + 3y - 4xy - xy^2 \) 
6. \( ax^3 - x^2z^2 - axyz + yz^3 \)
7. \( y^2 + 6y + 8 \) 
8. \( x^2 - x - 30 \) 
9. \( m^2 - 6m + 5 \) 
10. \( x^2 + 4x - 21 \) 
11. \( 2x^2 - 13x + 20 \) 
12. \( 5x^2 + 22xy + 8y^2 \) 
13. \( 56p^2 - 13pq - 3q^2 \) 
14. \( r^4 + r^2 - 2 \)
15. \( x^2 - 4x + 4 \) 
16. \( 4m^2 - 12m + 9 \) 
17. \( 36 - 12y + y^2 \) 
18. \( 4m^2 - 12mn + 9n^2 \) 
19. \( x^2 - 4xy + 4y^2 \) 
20. \( m^2 - 6mn + 9n^2 \)

Activity 6.3
1. \( 4(a + 2b) \) 
2. \( 2(1 - 3k) \) 
3. \( x(x + y) \) 
4. \( q(p - 3q) \) 
5. \( x(xy - 1) \) 
6. \( z(xy + z^2) \) 
7. \( 3x(1 + 2x) \) 
8. \( 5x(1 - 3xy) \) 
9. \( 4x^2(2 - xy) \) 
10. \( -5x(3y - 11z) \) 
11. \( 2b(c - 3bc + 6b^2) \) 
12. \( x^3(x^3 - 4 + x^2) \) 
13. \( -2x^2(4x - 3 + 7x^3) \) 
14. \( 5xy^2(2x^2y^2 - x - 2y) \)

Activity 6.4
1. \( (x + 6)(x - 6) \) 
2. \( (m + 8)(m - 8) \) 
3. \( (y + 1)(y - 1) \) 
4. \( (10 + x)(10 - x) \) 
5. \( (p + q)(p - q) \) 
6. \( (2n + 7)(2n - 7) \) 
7. \( (1 - k)(1 + k) \) 
8. \( (5 + 3t)(5 - 3t) \)
9. \(7(1 + 2y)(1 - 2y)\) 
10. \(4(x + 2)(x - 2)\) 
11. \(y(1 + y)(1 - y)\) 
12. \(6(2x + 5y)(2x - 5y)\) 
13. \(3(x + 5)(x - 5)\) 
14. \(3a(3b + c)(3b - c)\)

**Activity 6.5**

1. \((x + 1)(x + 2)\) 
2. \((x + 1)(x - 2)\) 
3. \((x - 3)(x - 4)\) 
4. \((x + 5)(x - 3)\) 
5. \((x + 8)(x - 3)\) 
6. \((x + 1)(2x + 7)\) 
7. \((3x - 1)(x + 4)\) 
8. \((5x + 2)(x - 3)\)

**Activity 6.6**

1. \(\frac{x + 2}{x}\) 
2. \(\frac{3(t - 8)}{2t(t - 7)}\) 
3. \(-(x + y)^2\) 
4. \(p\) 
5. \(\frac{2(a + 4)}{(a - 4)(a + 2)}\) 
6. \(\frac{z(z - 2)}{3(z - 4)(z - 3)}\) 
7. \(\frac{2x}{x^2 - 1}\) 
8. \(\frac{t + 2}{t(t + 1)}\) 
9. \(\frac{p^2 + 2p + 2}{(p + 2)(p + 1)}\) 
10. \(\frac{4a}{a^2 - 1}\) 
11. \(\frac{2x - 3}{(x - 2)(x + 1)(x - 1)}\) 
12. \(\frac{1}{1 - t}\)

**Activity 6.7**

1. \(x = 1\) 
2. \(x = 3\) or \(-5\) 
3. \(x = 0\) or \(14\) 
4. \(x = 7\) or \(-4\) 
5. \(p = 10\) or \(-10\) 
6. \(k = 1\) or \(5\) 
7. \(x = 2\) or \(-5\) 
8. \(c = 2.4\) or \(0\) 
9. \(x = 2\) or \(-2\) 
10. \(y = \frac{1}{3}\) or \(-\frac{1}{3}\) 
11. \(x = 0\) or \(x = 7\) 
12. \(v = 0\) or \(18\)
Activity 6.8
1. $x = 4$ or $-1.5$
2. $x = 2$ or $1$
3. $t = 0.64$ or $-0.39$
4. $p = 0.67$ or $-0.25$
5. $b = -0.38$ or $-2.62$
6. $k = 13.08$ or $0.92$

Activity 6.9
1. 7 hours
2. 55 m, 30 m
3. $\approx 3$ m
4. 4 cm, 5 cm
5. 30 m by 16 m and 40 m by 12 m
6. 33.3 km/h and 36.0 km/h (1 dp)

Activity 6.10
1. 
2. 
3. 
4. 
Activity 6.11

1. c) 

2. b) 

3. $x = 8$ and $y = -12$ or 
   $x = 1$ and $y = -3$

4. Solutions are: $(2, -1)$ and $(-1, 2)$
Section 7 – Statistics

7.1 Summation notation

In statistics we frequently have to add scores. The symbol \( \Sigma \) is used to designate a sum or total. The actual theory behind the symbol and its use with subscripts and superscripts is not particularly relevant for us. We have only use for expressions such as \( \Sigma X \), meaning the sum of all scores and \( \Sigma X^2 \) meaning the sum of the squares of all scores.

Consider the following example.

Example 1  Jack scored the following number of points per match during the basketball season.

<table>
<thead>
<tr>
<th>Match</th>
<th>Free Kicks (X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>28</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

Letting \( X \) be the number of points per match, calculate \( \Sigma X \) and \( \Sigma X^2 \)

\[
\Sigma X = 20 + 32 + 12 + 23 + 14 + 23 + 11 + 9 + 28 + 12 + 22 + 16 + 4 + 16 + 36 + 22 + 21 + 19 + 15 + 21
\]

\[
\Sigma X^2 = 20^2 + 32^2 + 12^2 + 23^2 + 14^2 + 23^2 + 11^2 + 9^2 + 28^2 + 12^2 + 22^2 + 16^2 + 4^2 + 16^2 + 36^2 + 22^2 + 21^2 + 19^2 + 15^2 + 21^2
\]

\[
\Sigma X = 376
\]

\[
\Sigma X^2 = 8212
\]

Note that \( \Sigma X^2 \) is not the same as \( (\Sigma X)^2 \) because this last value stands for the ‘square of the sum of all scores’ which in the above example is

\[
376^2 = 141376
\]

Activity 7.1

1. The Deaglers Footy team coach kept track of the number of free kicks awarded against the team. Calculate \( \Sigma X \) and \( \Sigma X^2 \).

<table>
<thead>
<tr>
<th>Match</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free Kicks (X)</td>
<td>10</td>
<td>12</td>
<td>9</td>
<td>14</td>
<td>12</td>
<td>15</td>
<td>8</td>
<td>10</td>
<td>13</td>
<td>12</td>
</tr>
</tbody>
</table>

2. Consider the following scores: 1 2 2 3 4 5 5 7 8 and 10.

a) Calculate \( \Sigma X \), \( \Sigma X^2 \), \( (\Sigma X)^2 \)

b) Is there a difference between \( \Sigma X - 1 \) and \( \Sigma (X - 1) \)?

Check your answers at the end of this Section.
7.2 Grouping data

When a large number of single (discrete) data is analysed, it makes sense to arrange data into a more compact form without obscuring the essential information that is contained in the data. The frequency distribution is a tool to achieve this. In a frequency distribution, data items are grouped into non-overlapping classes and the number of times an item appears in each class is recorded.

It is usually desirable to consider the following basic rules.

- The width of each class, the class interval, should be equal and of a convenient size eg 4, 5, 10, 20 etc.
- The number of classes should be more than 6 but less than 12 and there should be no gaps even if a particular class does not contain any items.

There are various ways to indicate the lower and upper values of a class:

1) using inequality signs eg $45 \leq x < 55$, $55 \leq x < 65$ etc
2) using words eg $45$ and under $55$, $55$ and under $65$ etc
3) using gaps eg $45–54$, $55–64$ etc.

The last method is best avoided because so-called real boundaries ($44.5–54.5$, $54.5–64.5$ etc) need to be calculated and used when drawing graphs. The first method is easiest. No further calculations are necessary and there is no confusion because there is no overlap. For example, 54.5 clearly belongs to the first class.

Later, for calculating the mean, we will make use of the class mid-point. This is the middle of a class. Thus the mid-point of $45 \leq x < 55$ is 50. We shall also make use of the cumulative frequency. The cumulative frequency of a class is the frequency in that class plus the sum of the frequencies of the classes below.

An example will clarify this concept.

Example 1 The following data were collected from a sample of 60 people who rented Perth hotel rooms in June last year. It shows the rental in dollars paid per day.

<table>
<thead>
<tr>
<th>60</th>
<th>44</th>
<th>38</th>
<th>35</th>
<th>53</th>
<th>64</th>
<th>49</th>
<th>73</th>
<th>29</th>
<th>87</th>
<th>90</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>62</td>
<td>45</td>
<td>53</td>
<td>42</td>
<td>57</td>
<td>74</td>
<td>54</td>
<td>76</td>
<td>45</td>
<td>36</td>
<td>87</td>
</tr>
<tr>
<td>55</td>
<td>65</td>
<td>84</td>
<td>62</td>
<td>79</td>
<td>45</td>
<td>59</td>
<td>69</td>
<td>25</td>
<td>76</td>
<td>98</td>
<td>56</td>
</tr>
<tr>
<td>30</td>
<td>75</td>
<td>50</td>
<td>53</td>
<td>61</td>
<td>58</td>
<td>86</td>
<td>52</td>
<td>55</td>
<td>76</td>
<td>87</td>
<td>48</td>
</tr>
<tr>
<td>33</td>
<td>80</td>
<td>30</td>
<td>42</td>
<td>55</td>
<td>76</td>
<td>67</td>
<td>43</td>
<td>76</td>
<td>54</td>
<td>65</td>
<td>64</td>
</tr>
</tbody>
</table>

Tabulate the data using intervals of $10$ starting with the class 20–30. Indicate the mid-point, the frequency (f) and the cumulative frequency (cf).
Mem 3012a

Apply mathematical techniques in a manufacturing, engineering or related environment

Class \((x)\) Score  | Mid-point | \(f\) | \(cf\)
---|---|---|---
\(20 \leq x < 30\) | 25 | 2 | 2
\(30 \leq x < 40\) | 35 | 7 | 9
\(40 \leq x < 50\) | 45 | 9 | 18
\(50 \leq x < 60\) | 55 | 15 | 33
\(60 \leq x < 70\) | 65 | 10 | 43
\(70 \leq x < 80\) | 75 | 9 | 52
\(80 \leq x < 90\) | 85 | 6 | 58
\(90 \leq x < 100\) | 95 | 2 | 60

Notice that the cumulative frequency is easily obtained by adding the frequency of a class to the cumulative frequency of the class before. Also note that the cumulative frequency of the last class should equal the total number of scores.

**Activity 7.2**

Prepare a frequency distribution for the results of 40 students in a mathematics test. The results are given to the nearest percent. Use a class interval of 10 and indicate mid-point, frequency \((f)\) and cumulative frequency \((cf)\).

| \(x\) | Mid-point | \(f\) | \(cf\)
|---|---|---|---
40 | 55 | 61 | 60 | 65
94 | 60 | 58 | 26 | 23
50 | 75 | 73 | 28 | 65
36 | 89 | 44 | 86 | 58
85 | 56 | 79 | 57 | 56
48 | 90 | 59 | 64 | 36
50 | 94 | 56 | 52 | 51
71 | 55 | 57 | 32 | 55

Check your answers at the end of this Section.
7.3 The mode

Very often we wish to describe a set of scores by using only one value. The value we select is the one most **typical** and most **representative** of the data. We often refer to it as the **central score** about which the other scores tend to bunch; it is therefore called a measure of **central tendency**.

One such score is the **mode** which is simply the most frequently occurring score. With **ungrouped** data we can establish the mode by inspection.

**Example** Find the mode of the following sets of data.

a) 2, 2, 2, 2  
   b) 3, 1, 3, 3, 1  
   c) 1, 1, 1, 2, 3, 4, 4, 6, 7, 8  
   d) 3, 3, 5, 9, 9  
   e) 1, 2, 3, 4, 5

By examining the data we obtain:

a) 2 (it is the only score)  
   b) 3  
   c) 1 (not a good central score)  
   d) 3 and 9 (a **bi-modal** distribution)  
   e) no mode

From consideration of the above example, we see that the mode’s effectiveness in representing every score is very limited. It is only used in specialised applications or to obtain a very quick estimate of central tendency.

When dealing with **grouped** data, we do not know the actual scores and hence the mode cannot be found. Sometimes the **modal class** is asked for and this is the class in the distribution with the highest frequency.

**Activity 7.3**

Find the mode of the following sets of data if it exists.

a) 1, 3, 4, 6, 7  
   b) 3, 1, 1, 3, 1, 3, 1, 1, 1  
   c) 1, 1, 2, 3, 4, 4, 4, 6, 7, 8  
   d) 1, 1, 2, 2, 2, 3,3, 3, 3

**Check your answers at the end of this Section.**
7.4 The median

The median is the 50% cut-off point value i.e. the value below and above which 50 per cent of scores lie. It is a far more useful measure because it is not influenced by extreme scores, generally called outliers. For this reason it is used, for example, by the Real Estate Institute which frequently publishes the median prices of houses in various cities.

For ungrouped data the median is the middle score when scores are in order. If the number of scores is even, the two scores in the middle are averaged.

Example 1  Find the median of
a)  2, 3, 4, 5, 6, 6, 7, 8, 9  
    b)  2, 3, 4, 5, 5, 6, 7, 8, 9, 10

There are an odd number of scores in a) and the median is 6. In b) on the other hand, the number of scores is even and both 5 and 6 are in the middle. Hence the median is 5.5.

It is frequently necessary to order the scores first before the median can be found.

Example 2  The number of customers who made inquiries at the information desk of a large department store was recorded for each Saturday morning over a period of ten weeks. The numbers were:

47  55  38  43  37  92  37  49  52  50

Calculate the median.

Arranging the scores in order: 37   37   38   43   47   49   50   52   55   92

The median is \( \frac{1}{2} \) of (47 + 49) = 48.

The median for grouped data can be approximated using a formula that looks quite complicated but is easy to use after some practice.

\[
\text{Median} (Q_2) = L + \left( \frac{\frac{n}{2} - C}{f} \right) (i)
\]

where

- \( L = \) the real lower limit of the frequency class in which the median occurs (median class)
- \( n = \) the number of scores in the data
- \( C = \) the cumulative frequency of the class preceding the median class
- \( f = \) the frequency of the median class
- \( i = \) the size of each class interval

The method is illustrated in the following example.
Example 3  Use the Hotel rental data in the example of section 7.2, reproduced on the right to calculate the median.

Here
\[ L = 50 \]
\[ n = 60 \]
\[ C = 18 \]
\[ f = 15 \]
\[ i = 10 \]

<table>
<thead>
<tr>
<th>Class ((x)) Score</th>
<th>(f)</th>
<th>(cf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 (\leq x &lt; 20)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20 (\leq x &lt; 30)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>30 (\leq x &lt; 40)</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>40 (\leq x &lt; 50)</td>
<td>9</td>
<td>18</td>
</tr>
<tr>
<td>50 (\leq x &lt; 60)</td>
<td>15</td>
<td>33</td>
</tr>
<tr>
<td>60 (\leq x &lt; 70)</td>
<td>10</td>
<td>43</td>
</tr>
<tr>
<td>70 (\leq x &lt; 80)</td>
<td>9</td>
<td>52</td>
</tr>
<tr>
<td>80 (\leq x &lt; 90)</td>
<td>6</td>
<td>58</td>
</tr>
<tr>
<td>90 (\leq x &lt; 100)</td>
<td>2</td>
<td>60</td>
</tr>
</tbody>
</table>

Median \((Q_2)\) = \(50 + \left( \frac{60 - 18}{15} \times 10 \right)\)

= 58

Activity 7.4

1. The waiting times (in minutes) for doctor’s treatment at a Medical Centre for 15 patients are:

14, 28, 36, 15, 29, 16, 9, 40, 16, 21, 36, 17, 4, 15, 22

Calculate the median waiting time.

2. A national business school administered an examination to their 60 final year students. The results were summarised in the following grouped frequency distribution:

<table>
<thead>
<tr>
<th>Test score ((x))</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 (\leq x &lt; 50)</td>
<td>6</td>
</tr>
<tr>
<td>50 (\leq x &lt; 60)</td>
<td>12</td>
</tr>
<tr>
<td>60 (\leq x &lt; 70)</td>
<td>14</td>
</tr>
<tr>
<td>70 (\leq x &lt; 80)</td>
<td>19</td>
</tr>
<tr>
<td>80 (\leq x &lt; 90)</td>
<td>6</td>
</tr>
<tr>
<td>90 (\leq x &lt; 100)</td>
<td>3</td>
</tr>
<tr>
<td>TOTAL</td>
<td>60</td>
</tr>
</tbody>
</table>

Find the median result.

Check your results at the end of this Section.
7.5 The mean

The most widely used measure of central tendency is the mean. This measure is sometimes referred to as average but in mathematics there are many types of averages. In fact both the mode and median are averages. Thus the term is best avoided.

The mean is the balancing point and is the only central tendency measure that uses all scores in calculating it. In fact, it is obtained by adding all the scores and then dividing by the number of scores. If \( X \) is used to designate a score then \( \overline{X} \) (pronounced ‘\( X \) bar’) is used for the mean of all scores \( X \). Also \( n \) is generally used for the total number of scores. This leads to the formula

\[
\overline{X} = \frac{\sum X}{n}
\]

**Example 1** Calculate the mean of 4 5 6 7 3 4 5 6 2 and 5.

Here \( \sum X = 47 \) and \( n = 10 \). Hence \( \overline{X} = \frac{47}{10} = 4.7 \)

If in the previous example, the scores had been, say, the number of people entering a store per minute, the answer 4.7 would not have been appropriate. The answer would then best be given as 5 people.

When a score repeats, the easiest way to calculate the mean is by using a frequency table.

**Example 2** Consider the numbers of runs scored by a batsman in his innings in which he faced 20 balls:

\[ 0 \ 2 \ 3 \ 4 \ 1 \ 0 \ 2 \ 0 \ 2 \ 1 \ 1 \ 4 \ 6 \ 4 \ 0 \ 0 \ 4 \ 4 \ 4 \ 1 \]

Calculate his ‘strike rate’ per ball faced.

We organize the data in a table and include an extra column. This has been done because the result of adding, for example, 7 fours is the same as multiplying 7 by 4.

<table>
<thead>
<tr>
<th>Runs (X)</th>
<th>Frequency ( f )</th>
<th>Sub-total ( fX )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

\[
\sum f = 20 \quad \sum fX = 43
\]
Section 7  Statistics

There are 20 scores ie \( n = 20 \). In the table this number is indicated by \( \sum f \).

Similarly, the total of the scores, is now replaced by \( \sum fX \)

Then \( \bar{X} = \frac{\sum fX}{\sum f} = \frac{43}{20} = 2.15 \) runs

When data are grouped into classes, we don’t know the individual scores. For example, in a table the frequency of a class 8–12 may be 7 but how are the scores distributed in this class? Do they all lie near 8, near 12 or somewhere near the middle? Our best estimate of each individual score would be 10, the class mid-point. This is mathematically the same as assuming that the scores are evenly distributed in the class.

It is indeed class mid-points that are used in calculating the mean for grouped data. This is illustrated in the following example where again we use the hotel data.

**Example 3** Calculate the mean rate per hotel room by letting \( X \) be the mid-point of each class.

<table>
<thead>
<tr>
<th>Class ≤ Score &lt;</th>
<th>( f )</th>
<th>( X )</th>
<th>( fX )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20–30</td>
<td>2</td>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>30–40</td>
<td>7</td>
<td>35</td>
<td>245</td>
</tr>
<tr>
<td>40–50</td>
<td>9</td>
<td>45</td>
<td>405</td>
</tr>
<tr>
<td>50–60</td>
<td>15</td>
<td>55</td>
<td>825</td>
</tr>
<tr>
<td>60–70</td>
<td>10</td>
<td>65</td>
<td>650</td>
</tr>
<tr>
<td>70–80</td>
<td>9</td>
<td>75</td>
<td>675</td>
</tr>
<tr>
<td>80–90</td>
<td>6</td>
<td>85</td>
<td>510</td>
</tr>
<tr>
<td>90–100</td>
<td>2</td>
<td>95</td>
<td>190</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td>60</td>
<td></td>
<td>3550</td>
</tr>
</tbody>
</table>

Then \( \bar{X} = \frac{\sum fX}{\sum f} = \frac{3550}{60} = 59.1666\ldots \)

Thus the mean rent is $59.17

The next example requires a little thinking.
Example 4  A basketball player scored on average 16.2 points per game over 15 games. How many points does she need to score in her next game to improve her average to 17 points?

The total the basketball player has scored in 15 games is $15 \times 16.2 = 243$ points.

If she wants to average 17 over 16 games, she needs to have a total of $16 \times 17 = 272$ points.

Thus she needs to score $272 - 243 = 29$ points in game 16.

Before finishing the topics on measures of central tendency, it should be pointed out that there is no such thing as a ‘best’ or ‘all-purpose’ measure. The choice of which to use in a given statistical situation must be determined by the nature of the data and the purposes to be served.

For example, in an earlier illustration, it was mentioned that it is not sensible to talk about 4.7 shoppers per minute entering a shop. Apart from the non sensible value, a shopkeeper would probably be more interested in the mode.

In most situations the mean is preferred, essentially because it uses all scores. However, the mean has a severe disadvantage; it is influenced by outliers. For example, if we calculated the mean age of 10 students who are 4, 4, 4, 5, 5, 6, 6, 6 and 16 years old, it would be 6.1 years. In this case 9 of the 10 students are below the mean age. The extreme value of 16 has distorted the picture; without it the mean of the remaining ages is 5.

Activity 7.5

1. Sir Donald Bradman’s scores in his first 12 completed innings in Test cricket were:

   $18, 79, 112, 40, 58, 123, 8, 131, 254, 1, 334, 14$

   Calculate his mean score, commonly called ‘average’.

2. Two dice were thrown and the total was recorded with the following results:

   $2, 11, 8, 7, 5, 6, 5, 4, 8, 9, 4, 5, 7, 7, 8, 11, 8, 7, 6, 12, 11, 2, 11, 10, 7, 7, 8, 3, 4, 10$

   Obtain the mean.

3. A personnel manager is interested in how long is spent by the factory employees in travelling from home to work each day. A random sample of 99 employees is chosen and their travel times (in minutes) are shown, on the next page, in the form of a grouped frequency distribution:
### Section 7 Statistics

#### Time taken (minutes) Number of employees

<table>
<thead>
<tr>
<th>Time taken (minutes)</th>
<th>Number of employees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 and under 10</td>
<td>3</td>
</tr>
<tr>
<td>10 and under 20</td>
<td>19</td>
</tr>
<tr>
<td>20 and under 30</td>
<td>25</td>
</tr>
<tr>
<td>30 and under 40</td>
<td>35</td>
</tr>
<tr>
<td>40 and under 50</td>
<td>14</td>
</tr>
<tr>
<td>50 and under 60</td>
<td>3</td>
</tr>
</tbody>
</table>

Calculate the mean time, correct to the nearest second.

4. A person scored an average of 65% on 4 completed tests. She wants to improve her average to 70%. How much should she score in Test 5?

Check your answers at the end of this Section.

### 7.6 The range and inter-quartile range

Consider two sets of data

- A: 99, 100, 101
- B: 0, 100, 200

For both sets, the mean and median are 100.

Can we say that since 100 best represents the data, the two data are alike in all respects? Study the data again. You can see that the scores in set B are spread farther apart than the scores of A. The mean, mode or median do not tell us anything about the spread (other words are scatter, dispersion or variability) of the scores. Thus we need another measure, a measure of dispersion, to indicate the spread or scatter.

One measure of dispersion is the range. It is the difference between the greatest and least score of the data. The range certainly has its uses but it is limited in that to calculate it, only two scores are used. This leads to situations in which two sets of scores can have the same range but yet have a very different dispersion. Look at the following sets. Both have a range of 29.

- C: 1, 5, 10, 15, 20, 25, 30
- D: 1, 22, 23, 24, 25, 26, 27, 30

To calculate the range, the scores need to be put in order first.

**Example 1** Calculate the range of these hourly wages:

$20, $31, $10, $21, $11, $19, $36, $9

The wages in order are:

9 10 11 19 20 21 31 36

Hence the range is $36 − $9 = $27
Sometimes use is made of the Inter-Quartile Range (IQR). This the range of the middle 50 per cent of scores. In formula form:

\[ IQR = Q_3 - Q_1 \]

where \( Q_1 \) = the lower quartile and \( Q_3 \) = the upper quartile.

These quartiles are defined in a similar way to the median which is the middle quartile \( Q_2 \).

For discrete data the IQR is not too difficult to calculate especially if the number of scores is divisible by 4. Sometimes we have to average.

Consider this example.

**Example 1** Calculate the inter-quartile range of the hourly wages:

$20, \ $31, \ $10, \ $21, \ $11, \ $19, \ $36, \ $9

Arranging the numbers order:

9   10    11   19   20  21   31  36

lower quartile = \( Q_1 \) = 10.5  
upper quartile = \( Q_3 \) = 26

The IQR = 26 − 10.5 = $15.50

**Activity 7.6**

1. The number of absentees from a Statistics class was recorded over a 16-week period:

1   2   4   3   5   2   4   1   5   2   1   4   7   6   1   8

Calculate the:

a) range  
b) inter-quartile range

Check your answers at the end of this Section.
7.7 The standard deviation

In contrast to the range, the standard deviation is a measure of dispersion that takes account of all the scores in the distribution.

The standard deviation makes use of so-called deviations from the mean. These values show how far each score is away from the mean of all scores. These values are then squared and averaged before the square root is taken. In formula form and using the symbol S:

\[ S = \sqrt{\frac{\sum (X - \bar{X})^2}{n}} \]

This formula looks complicated to use but if you use a table and calculate S using the following steps, the process is not too difficult.

• Set up a table with column headings \( X \), \( X - \bar{X} \) and \( (X - \bar{X})^2 \).
• Calculate the mean.
• Complete the table.
• Obtain the sums and check that the second column adds to 0.
• Average the total in the third column and take the square root.

Let us work an example.

**Example** Calculate the standard deviation of 15, 19, 17, 12, 9, 15, 16, 13.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( (X - \bar{X}) )</th>
<th>( (X - \bar{X})^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>15 – 14.5 = +0.5</td>
<td>0.5² = 0.25</td>
</tr>
<tr>
<td>19</td>
<td>4.5</td>
<td>20.25</td>
</tr>
<tr>
<td>17</td>
<td>2.5</td>
<td>6.25</td>
</tr>
<tr>
<td>12</td>
<td>-2.5</td>
<td>6.25</td>
</tr>
<tr>
<td>9</td>
<td>-5.5</td>
<td>30.25</td>
</tr>
<tr>
<td>15</td>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>16</td>
<td>1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>13</td>
<td>-1.5</td>
<td>2.25</td>
</tr>
<tr>
<td>Total</td>
<td>116</td>
<td>0</td>
</tr>
</tbody>
</table>

The working in the first line is shown above.

Note that the totals (sums) are

\[ \sum X = 116, \quad \sum (X - \bar{X}) = 0, \quad \sum (X - \bar{X})^2 = 68 \]

Firstly find the mean.

\[ \bar{X} = \frac{\sum X}{n} = \frac{116}{8} = 14.5 \]
Apply mathematical techniques in a manufacturing, engineering or related environment

MEM30012A

Activity 7.7

1. Calculate the standard deviation of 100, 150, 200 and 300. Use 2 decimal place accuracy.

2. Calculate, correct to 4 decimal places, the standard deviation of sets A and B at the beginning of section 7.6.

3. Calculate $S$ for the following dollar amounts $4$, $6$, $10$, $11$, $14$.
   Round to the nearest cent.

Check your answers at the end of this Section.

7.8 The alternative formula for standard deviation

Using relatively simple algebra, an alternative formula for the standard deviation can be derived. This formula should always be used since it involves a table with only two columns and hence reduces the workload extensively.

The formula is

$$S = \sqrt{\frac{\sum X^2}{n} - \overline{X}^2}$$

and a table with scores, $X$, and squares of scores, $X^2$, is required.

Let us do the example in the previous section again.
Example 1  Calculate the standard deviation of 15, 19, 17, 12, 9, 15, 16, 13

<table>
<thead>
<tr>
<th>$X$</th>
<th>$X^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>19</td>
<td>361</td>
</tr>
<tr>
<td>17</td>
<td>289</td>
</tr>
<tr>
<td>12</td>
<td>144</td>
</tr>
<tr>
<td>9</td>
<td>81</td>
</tr>
<tr>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>16</td>
<td>256</td>
</tr>
<tr>
<td>13</td>
<td>169</td>
</tr>
</tbody>
</table>

**Total** 116  1750

Note that the totals (sums) are:
\[ \sum X = 116, \quad \sum X^2 = 1750 \]

Then \[ \bar{X} = \frac{\sum X}{n} = \frac{116}{8} = 14.5 \]

and \[ S = \sqrt{\frac{\sum X^2}{n} - \bar{X}^2} = \sqrt{\frac{1750}{8} - 14.5^2} \]

\[ = \sqrt{218.75 - 210.25} \]

\[ = \sqrt{8.5} \]

\[ = 2.9 \text{ (one dp)} \]

Thus, as expected, we obtain the same result as before.

The alternative formula is especially useful when we deal with frequency data.

The formula involving frequencies becomes \[ S = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{X}^2} \] and requires 4 columns.

Of course \( \bar{X} \) is obtained from \[ \bar{X} = \frac{\sum fx}{\sum f} \]

Note that \( \sum fX^2 \) is not the same as \( (\sum fX)^2 \) and must be calculated separately.
The following example illustrates the use of this formula and how a value such as 1200 is obtained in the table below.

**Example 2** Use frequencies to calculate the standard deviation for the scores

20, 20, 19, 21, 21, 18, 20, 22, 23, 17

In table format:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$f$</th>
<th>$fX$</th>
<th>$fX^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>1</td>
<td>17</td>
<td>289</td>
</tr>
<tr>
<td>18</td>
<td>1</td>
<td>18</td>
<td>324</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>19</td>
<td>361</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>60</td>
<td>1200</td>
</tr>
<tr>
<td>21</td>
<td>2</td>
<td>42</td>
<td>882</td>
</tr>
<tr>
<td>22</td>
<td>1</td>
<td>22</td>
<td>484</td>
</tr>
<tr>
<td>23</td>
<td>1</td>
<td>23</td>
<td>529</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>10</td>
<td>201</td>
<td>4069</td>
</tr>
</tbody>
</table>

The totals (sums) are $\sum f = 10$, $\sum fX = 201$, $\sum fX^2 = 4069$

Then $\bar{X} = \frac{\sum fX}{\sum f} = \frac{201}{10} = 20.1$

and $S = \sqrt{\frac{\sum fX^2}{\sum f} - \bar{X}^2} = \sqrt{\frac{4069}{10} - 20.1^2} = \sqrt{406.9 - 404.01}

$= \sqrt{2.89} = 1.7$

As with the mean, when data is grouped in classes, the mid-point of each class is used to calculate the standard deviation.
Example 3  Find the standard deviation of the hourly rates earned by 200 McJack’s workers.

<table>
<thead>
<tr>
<th>Class (x) Hourly Rate</th>
<th>Frequency ( f )</th>
<th>Mid-point ( X )</th>
<th>( fX )</th>
<th>( fX^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 ≤ x &lt; 13</td>
<td>5</td>
<td>12</td>
<td>60</td>
<td>720</td>
</tr>
<tr>
<td>13 ≤ x &lt; 15</td>
<td>7</td>
<td>14</td>
<td>98</td>
<td>1372</td>
</tr>
<tr>
<td>15 ≤ x &lt; 17</td>
<td>22</td>
<td>16</td>
<td>352</td>
<td>5632</td>
</tr>
<tr>
<td>17 ≤ x &lt; 19</td>
<td>35</td>
<td>18</td>
<td>630</td>
<td>11340</td>
</tr>
<tr>
<td>19 ≤ x &lt; 21</td>
<td>53</td>
<td>20</td>
<td>1060</td>
<td>21200</td>
</tr>
<tr>
<td>21 ≤ x &lt; 23</td>
<td>46</td>
<td>22</td>
<td>1012</td>
<td>22264</td>
</tr>
<tr>
<td>23 ≤ x &lt; 25</td>
<td>22</td>
<td>24</td>
<td>528</td>
<td>12672</td>
</tr>
<tr>
<td>25 ≤ x &lt; 27</td>
<td>10</td>
<td>26</td>
<td>260</td>
<td>6760</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>200</strong></td>
<td></td>
<td><strong>4000</strong></td>
<td><strong>81,960</strong></td>
</tr>
</tbody>
</table>

The totals (sums) are \( \sum f = 200, \sum fX = 4000, \sum fX^2 = 81,960 \)

Then \[ \bar{X} = \frac{\sum fX}{\sum f} = \frac{4000}{200} = 20 \]

and \[ S = \sqrt{\frac{\sum fX^2}{\sum f} - \bar{X}^2} = \sqrt{\frac{81,960}{200} - 20^2} = \sqrt{409.8 - 400} \]

\[ = \sqrt{9.8} = 3.13... \]

Thus the standard deviation of the wages is $3.13.

Activity 7.8

1. The results of a mathematics test involving a class of 34 students were recorded as follows:

<table>
<thead>
<tr>
<th>Mark</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of students</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Calculate the standard deviation of the above distribution.
2. Marks awarded to 392 candidates at an examination are as below. Calculate S.

<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Frequency</th>
<th>Class Interval</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>11–15</td>
<td>6</td>
<td>36–40</td>
<td>96</td>
</tr>
<tr>
<td>16–20</td>
<td>12</td>
<td>41–45</td>
<td>54</td>
</tr>
<tr>
<td>21–25</td>
<td>30</td>
<td>46–50</td>
<td>37</td>
</tr>
<tr>
<td>26–30</td>
<td>3</td>
<td>51–55</td>
<td>19</td>
</tr>
<tr>
<td>31–35</td>
<td>77</td>
<td>56–60</td>
<td>8</td>
</tr>
</tbody>
</table>

Check your answers at the end of this Section.
Assessment 7

Set out your work to clearly show your reasoning at each stage of the solution. You are allowed to use a calculator to obtain totals.

1. A basketball player scored the following number of points in each of the 24 games he played for the Wilddogs last year.

   
<table>
<thead>
<tr>
<th>5</th>
<th>19</th>
<th>11</th>
<th>22</th>
<th>36</th>
<th>16</th>
<th>17</th>
<th>7</th>
<th>10</th>
<th>19</th>
<th>17</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td>12</td>
<td>17</td>
<td>13</td>
<td>25</td>
<td>10</td>
<td>16</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

   Calculate the:
   a) mode  
   b) median  
   c) mean  
   d) range  
   e) IQR  
   f) standard deviation

2. The time taken by 50 respondents to fill in a survey questionnaire was recorded, and the results tabulated as below:

<table>
<thead>
<tr>
<th>Time (x mins)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 ≤ x &lt; 1</td>
<td>1</td>
</tr>
<tr>
<td>1 ≤ x &lt; 2</td>
<td>3</td>
</tr>
<tr>
<td>2 ≤ x &lt; 3</td>
<td>10</td>
</tr>
<tr>
<td>3 ≤ x &lt; 4</td>
<td>24</td>
</tr>
<tr>
<td>4 ≤ x &lt; 5</td>
<td>8</td>
</tr>
<tr>
<td>5 ≤ x &lt; 6</td>
<td>4</td>
</tr>
</tbody>
</table>

   Determine the:
   a) modal class  
   b) median  
   c) mean  
   d) standard deviation
Answers to activities

Activity 7.1
1. 115 and 1 367

2. a) 47, 297 and 2 209
   b) Yes: 46 vs 37

Activity 7.2

<table>
<thead>
<tr>
<th>Class ((x)) Score</th>
<th>Mid-point</th>
<th>(f)</th>
<th>(cf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 (\leq x &lt; 30)</td>
<td>25</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>30 (\leq x &lt; 40)</td>
<td>35</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>40 (\leq x &lt; 50)</td>
<td>45</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>50 (\leq x &lt; 60)</td>
<td>55</td>
<td>15</td>
<td>24</td>
</tr>
<tr>
<td>60 (\leq x &lt; 70)</td>
<td>65</td>
<td>6</td>
<td>30</td>
</tr>
<tr>
<td>70 (\leq x &lt; 80)</td>
<td>75</td>
<td>4</td>
<td>34</td>
</tr>
<tr>
<td>80 (\leq x &lt; 90)</td>
<td>85</td>
<td>3</td>
<td>37</td>
</tr>
<tr>
<td>90 (\leq x &lt; 100)</td>
<td>95</td>
<td>3</td>
<td>40</td>
</tr>
</tbody>
</table>

Activity 7.3
a) No mode
b) 1
c) 4
d) 3

Activity 7.4
1. 17 minutes
2. 68.57

Activity 7.5
1. 97.67
2. 7.1
3. 29.75 minutes
4. 90%

Activity 7.6
a) 7
b) 3.5
Activity 7.7
1. 73.95
2. Set A: 0.8165  Set B: 81.6497
3. $3.58

Activity 7.8
1. 2.10 (2 decimal places)
2. 9.36 (2 decimal places)
APPLY MATHEMATICAL TECHNIQUES IN A
MANUFACTURING, ENGINEERING OR RELATED
ENVIRONMENT
MEM30012A

Learner’s Guide

DESCRIPTION
This MEM30012A unit applies the concepts of mathematics to appropriate and simple engineering situations within various areas of engineering expertise. The skills learnt will be used in more advanced mathematical subjects so it is of great importance that the student becomes proficient with all the techniques covered. The seven topics included are: Arithmetic, Algebra, Geometry, Trigonometry, Linear Functions, Quadratic Functions and Statistics.

EDITION
• Second Edition

CATEGORY
• Science and Mathematics

TRAINING PACKAGE
• Metal and Engineering

COURSE / QUALIFICATION
• Diploma in Engineering - Technical (Mechanical)

UNIT OF COMPETENCY
• MEM30012A  Apply Mathematical Techniques in a Manufacturing, Engineering, or Related Environment

RELATED PRODUCTS
• ENG1146  Use Quadratic, Exponential, Logarithmic and Trigonometric Functions and Matrices