USE QUADRATIC, EXPONENTIAL, LOGARITHMIC AND TRIGONOMETRIC FUNCTIONS AND MATRICES
EDX140
Learner’s Guide
Engineering
Use Quadratic, Exponential, Logarithmic and Trigonometric Functions and Matrices

EDX140

Learner’s Guide
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Use quadratic, exponential, logarithmic and trigonometric functions and matrices.
Introduction

Welcome to this unit. In this unit you will learn various ‘pre-calculus’ mathematical techniques relating to algebra, trigonometry and matrix algebra. These techniques may be described as essential knowledge in order to study technology-oriented subjects at Diploma level, such as fluid dynamics and statics. If you want to succeed in an engineering-related field beyond Diploma level in which calculus is essential, you will need to be thoroughly familiar with the material that will be covered.

The time needed to complete this unit is 60 nominal hours. Some students will require the full 60 hours, whereas others may have existing skills which will enable them to complete the unit sooner.

Prerequisites

The formal prerequisite for this unit is EDX130 Mathematics at technician level. A subject with an equivalent difficulty level may be substituted. You should not attempt this unit if you do not have a sound knowledge of elementary algebra and trigonometry. You should also know how to use a scientific calculator. Knowing how to use a graphics calculator is a distinct advantage.

How to use this guide

This Learner’s Guide is divided into seven Sections, covering the fourteen elements of competence.

Each Section is divided into a number of topics. Every topic ends with an activity for you to try. The answers to these activities are provided at the end of the Section you are working on.

To make full use of this guide, carefully read and study the notes and worked examples for a topic and then complete the activity.

Check your answers and, provided that you have made no errors or only minor errors, move on to the next topic.

If you are not happy with your results on a particular activity, reread the notes and work through the examples again. Reading another text may help. If necessary, contact your tutor for advice.

When you have finished all the topics in a Section, revise your work and then complete the assignment for the Section. If you study this unit by correspondence, send this assignment to your tutor and then commence the next Section.

Helpful tips

Make sure you understand all the material in this guide along with any other texts that you are using. Please consider the following if you are unsure about the content.

• Talk to or make contact with your facilitator.
• Research books, videos and websites.
• Talk to other students.
Resources

Texts
There is no set text for this unit. This Learner’s Guide incorporates all instructional material relating to the unit.

Internet
There are many mathematics-related resources online. Useful information may be obtained by typing a relevant word such as ‘radian measure’ into a search engine.

Calculator
Although some parts of the unit can be done without a calculator (the first Section, for example), you will need a scientific calculator for most of the other Sections.

Should you have a graphics calculator, use it for checking answers. In tests and assignments you will always be required to show full working out in order to obtain full marks.
Elements of competence

Note: In this learning guide, the 14 elements are grouped into seven Sections.

1. Polynomials
2. Functions
3. Indices
4. Logarithms
5. Trigonometry part 1
6. Trigonometry part 2
7. Matrices.

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<tr>
<td>1</td>
<td>1</td>
<td>Solve practical problems using polynomials. Polynomial expressions are manipulated and simplified using addition, subtraction, multiplication and factoring in the correct order. The distributive law is used in the manipulation and simplification of polynomial expressions. Scientific notation is used to represent numbers. Trinomials are factored using trial and error, the difference between two squares and other methods. Quadratic equations are solved using the factoring and complete the square methods. Quadratic equations are solved using the quadratic formula. Rational binomial and trinomial algebraic expressions are manipulated and simplified. Quadratic equations are graphed and sketched in order to determine solutions to practical vocational problems.</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>Solve vocational mathematics problems using indices. Exponential expressions containing positive indices are simplified using the index laws. Exponential problems containing negative, fractional and zero indices are simplified. Expressions involving powers and roots are solved with a calculator. Numerical and literal expressions are expanded and simplified. Vocational formulae containing exponents are transposed.</td>
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<td>Element</td>
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<tr>
<td>3</td>
<td>2</td>
<td>Distinction can be made between a relation and a function. Given the equation of a function the graph can be sketched. Functions of the type ( y = mx + b ), ( y = \sqrt{x} ), ( y = \frac{a}{x} ) and ( y = a^x ) are solved. Calculations are performed using the typical functions of a graphics calculator. Quadratic functions are sketched from the defining rule and by completing the square, showing line of symmetry, ( x ) and ( y ) intercepts. Quadratic equations are solved graphically by using a graphics calculator. Equations are determined from graphs using systems consisting of a quadratic and linear equation are solved analytically. Systems consisting of a quadratic and linear equation are solved graphically using a graphics calculator. Non-routine vocational problems are solved using simple algebraic functions and their graphs.</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>Nonlinear data is transformed into linear data. The line of ‘best fit’ (regression) is drawn. The corresponding nonlinear formula is determined.</td>
</tr>
<tr>
<td>5</td>
<td>3, 4</td>
<td>Algebraic expressions are simplified using indices. Exponential equations are solved without using logarithms. The meaning of a logarithm as an exponent is described. Change of base formula and a calculator is used to evaluate logarithms. Logarithmic expressions are changed in their form. Exponential equations are solved using logarithms. Formulae involving logarithmic and exponential forms are transposed. The inverse of a function is defined. Exponential and logarithmic functions are graphed. The relationship between exponential and logarithmic functions is explained. Non-routine vocational problems are solved using exponents and logarithms.</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>Two simultaneous equations involving exponential, power and linear relationships are solved graphically. Growth and decay problems are solved graphically.</td>
</tr>
<tr>
<td>Element</td>
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<td>7</td>
<td>Vocational mathematics problems are solved by determining empirical laws for data related by either an exponential or a power law.</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>Vocational mathematical problems are solved using the unit circle definitions of trigonometric functions, graphs of circular functions and real number angular measure.</td>
<td>5, 6</td>
</tr>
<tr>
<td>9</td>
<td>Vocational mathematics problems are solved using the sine and/or the cosine rule.</td>
<td>5</td>
</tr>
<tr>
<td>Element</td>
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</table>
| 10 Vocational mathematics problems are solved using trigonometric identities. | 6 | Trigonometric expressions are simplified using the addition formulae.  
Trigonometric expressions are simplified using the double angle formulae.  
Trigonometric expressions are simplified using the sum-to-product formulae.  
Trigonometric expressions are simplified using the product-to-sum formulae.  
Trigonometric expressions are manipulated using the trigonometric ratios.  
Vocational problems are solved using trigonometric identities. |
| 11 Graph quadratic functions and solve quadratic equations. | 2 | Graphs of quadratic functions can be sketched and interpreted.  
The significance of the leading coefficient and the zeros can be shown.  
Quadratic equations can be solved using the quadratic formula.  
Simultaneous linear and quadratic equations can be solved algebraically and geometrically.  
Verbally formulated problems involving quadratic and linear equations can be interpreted and solved. |
| 12 Graph exponential and logarithmic functions and solve exponential and logarithmic equations. | 4 | Arithmetic and algebraic expression can be manipulated and simplified using the laws of indices and logarithms.  
The graphs of simple exponential and logarithmic functions can be graphed to show the behaviour for large and small values.  
Exponential and simple logarithmic equations can be solved using indices, logarithms, calculator and graphical techniques.  
Logarithms can be converted between bases, especially 10 and base e.  
Nonlinear functions (including exponential) can be transformed to linear forms and the data plotted.  
Lines of best fit can be drawn, data interpolated and constants estimated in suggested relationships.  
Verbally formulated problems involving growth and decay can be interpreted and solved. |
| 13 Graph trigonometric functions and solve trigonometric equations. | 6 | The graphs of simple trigonometric functions can be sketched showing the significance of amplitude, period and phase angle.  
Trigonometric expressions can be simplified using trigonometric identities. |
Assessment

The Elements and Performance Criteria for this unit are assessed by means of seven assessment tasks, one for each Section. These tasks are assignments consisting of a set of questions.

If you study this unit by correspondence, you should send the assignments to your facilitator. If you study this unit in a conventional classroom setting, your facilitator will inform you how this unit will be assessed. This may be by means of the assignments or by means of a series of tests and/or projects.
Use quadratic, exponential, logarithmic and trigonometric functions and matrices
Section 1 – Polynomials

1.1 Simplifying polynomials

An algebraic expression is a combination of numbers both variable and constant using the operations of arithmetic: addition, subtraction, multiplication, division and extraction of roots.

An example of an algebraic expression is:

\[ \sqrt{(x - 8)^3} \]

Note that an algebraic expression is NOT an equation. An equation must contain an equal sign to show the relationship between two algebraic expressions.

An example of an equation is:

\[ \sqrt{(x - 8)^3} = \frac{5}{7 + y^2} \]

An algebraic expression of the form \(3bx^2 + 5ax - 7\) contains three terms: \(3bx^2\), \(5ax\), and \(-7\). The first term consists of the factors 3, b, and \(x^2\). In the second term 5a is called the coefficient of \(x\).

Algebraic expressions with exactly one term are called monomials. Those having exactly two terms are binomials and those with exactly three terms are trinomials. In general, algebraic expressions with more than one term are called multinomials.

For example, \(12a^2x\) is a monomial, \(2x - 5\) is a binomial and \(3x^2 - 2x + 1\) is a trinomial.

A polynomial in \(x\) is an algebraic expression of the form:

\[ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \]

Where \(n\) is a non-negative integer, and the coefficients \(a_n, a_{n-1}, \ldots, a_1, a_0\) are constants with \(a_n\) not zero. The degree of the polynomial is \(n\).

Thus \(3x^3 + 7x^2 - 5x - 6\) is an example of a polynomial of degree 3.

We shall mainly be concerned with the manipulation of monomials, binomials and trinomials. Let us recall how algebraic expressions such as polynomials should be simplified.

Example 1 Simplify: \(4(x + 3) - 2(x - 1)\).

\[
4(x + 3) - 2(x - 1) = (4)(x) + (4)(3) - (2)(x) + (-2)(-1) \\
= 4x + 12 - 2x + 2 \\
= 2x + 14
\]
Example 2  Simplify: $x(x - 2y) + y(x + y)$.

$$x(x - 2y) + y(x + y) = x^2 - 2xy + yx + y^2$$
$$= x^2 - 2xy + xy + y^2$$
$$= x^2 - xy + y^2$$  (Change $yx$ to $xy$ so it can be combined with $-2xy$.)

Example 3  Simplify: $5k(3m + k) - 4m(4k + 3m)$.

$$5k(3m + k) - 4m(4k + 3m) = 15km + 5k^2 - 16mk - 12m^2$$
$$= 15km + 5k^2 - 16km - 12m^2$$
$$= 5k^2 - mk - 12m^2$$

Note that we usually rearrange the answer so that the first term does not have a negative sign.

Activity 1.1

Multiply and simplify:
1. $b(b + 4) - 5(b + 2)$   2. $p(p^2 - 7p) + 3(2p^2 + 4)$
3. $x^2(xy - 3y^2) - y^2(2x^2 - xy)$   4. $a(a + b - 3) - b(a - b + 2)$

1.2 Multiplying binomials

When two binomials are multiplied, the whole first binomial expression must be multiplied by each term in the second.

$$(a + b)(c + d) = (a + b)(c) + (a + b)(d)$$
$$= ca + cb + da + db$$
$$= ac + bc + ad + bd$$

We can produce this result by multiplying each term in the second bracket by each term in the first bracket.

$$(a + b)(c + d) = ac + ad + bc + bd$$

This is generally the way multiplication of binomials is carried out because it is quicker. It is sometimes called the **FOIL** (First, Outside, Inside Last) method.
Example 1 Multiply out: \((x + 2)(y + 4)\).

\[
(x + 2)(y + 4) = (x + 2)(y) + (x + 2)(4) = y(x + 2) + 4(x + 2) = xy + 2y + 4x + 8
\]

or

\[
(x + 2)(y + 4) = xy + 4x + 2y + 8
\]

Note that these results are the same if we rearrange the order of the terms. The important point is that the same terms appear in both expressions, even though not in the same order.

Example 2 Express without brackets: \((x - 2)(3y + 4)\).

\[
(x - 2)(3y + 4) = (x - 2)(3y) + (x - 2)(4) = (3y)(x - 2) + 4(x - 2) = 3xy - 6y + 4x - 8
\]

or

\[
(x - 2)(3y + 4) = 3xy + 4x - 6y - 8
\]

If both brackets contain the same letter or literal part, we end up with 'like' terms which should be combined.

This process is illustrated in the following examples.

Example 3 Simplify: \((m + 2)(m + 3)\).

\[
(m + 2)(m + 3) = m(m + 3) + 2(m + 3) = m^2 + 3m + 2m + 6
\]

We can see that the two middle terms are 'like terms', so we can combine them.

\[

∴ (m + 2)(m + 3) = m^2 + 5m + 6
\]

Example 4 Simplify: \((x + 1)(x - 2)\).

\[
(x + 1)(x - 2) = x(x - 2) + 1(x - 2) = x^2 - 2x + x - 2 = x^2 - x - 2
\]

Again the \((-2x)\) and \(x\) are 'like terms' and hence can be combined.
Example 5  Multiply: \((2x - 3)(x - 2)\).

\[
(2x - 3)(x - 2) = 2x(x - 2) - 3(x - 2)
\]

\[
= 2x^2 - 4x - 3x + 6
\]

\[
= 2x^2 - 7x + 6
\]

A special case of this procedure is considered in the last two examples.
You will remember that \(x^2 = x \cdot x\). So it follows that \((a + b)^2 = (a + b)(a + b)\).

Example 6  Simplify: \((2x + 4)^2\).

\[
(2x + 4)^2 = (2x + 4)(2x + 4)
\]

\[
= 2x(2x + 4) + 4(2x + 4)
\]

\[
= 4x^2 + 8x + 8x + 16
\]

\[
= 4x^2 + 16x + 16
\]

Example 7  Find the square of \((2x - 5)\) using the FOIL method.

The square of \((2x - 5) = (2x - 5)^2\)

\[
= (2x - 5)(2x - 5)
\]

\[
= 4x^2 - 10x - 10x + 25
\]

\[
= 4x^2 - 20x + 25
\]

Activity 1.2

Multiply and simplify if possible:
1. \((x + a)(y + b)\)
2. \((2x - 1)(4y + 3)\)
3. \((3p - q)(2p - 5q)\)
4. \((x^2 + 6)(5x - 1)\)
5. \((3 - xy)(4 + y)\)
6. \((x^2 - yz)(ax - z^2)\)

Multiply and simplify:
7. \((y + 2)(y + 4)\)
8. \((x - 6)(x + 5)\)
9. \((m - 1)(m - 5)\)
10. \((x + 7)(x - 3)\)
11. \((2x - 5)(x - 4)\)
12. \((5x + 2y)(x + 4y)\)
13. \((8p - 3q)(7p + q)\)
14. \((r^2 - 1)(r^2 + 2)\)

Express the following squares as trinomials.
15. \((x - 2)^2\)
16. \((2m - 3)^2\)
17. \((6 - y)^2\)
18. \((2m - 3n)^2\)
19. \((x - 2y)^2\)
20. \((-m + 3n)^2\)
1.3 Factorisation involving a common factor

Factorisation is the process of writing an expression in terms of factors.

For example the number 6 can be factorised and written as $2 \times 3$.

Polynomials may also be factorised. The process involves the inverse of the operation just discussed; instead of removing brackets, we now insert them.

Note that not all polynomials can be factored just as not all numbers (the prime numbers) can be factored.

The easiest type of factorisation involves a common factor, a factor that occurs in two or more terms in a polynomial.

Consider:

\[ a(b + c) = (a)(b) + (a)(c) \]
\[ = ab + ac \]

In reversing this multiplication, we factorise i.e.

\[ ab + ac = a(b + c) \]

Formally, the process is called **taking out** or **extracting** a common factor. Here $a$ is the common factor. The following examples will illustrate this process in more detail.

**Example 1** Factorise: $2x + 2y$.

Compare $2x + 2y$ with $ab + ac$.

We can see that $a = 2$, $b = x$, $c = y$.

So, if $ab + ac = a(b + c)$ then $2x + 2y = 2(x + y)$.

We can check by multiplying out: $2(x + y) = 2x + 2y$.

**Example 2** Factorise: $x^2 + 3x$.

\[ x^2 + 3x = (x)(x) + 3(x) = x(x + 3) \]

Check: $x(x + 3) = x^2 + 3x$ as required.

**Example 3** Factorise: $pq - 3pr$.

\[ pq - 3pr = (p)(q) + (p)(-3r) = p(q - 3r) \]

Check: $p(q - 3r) = pq - 3pr$ as required.
Example 4  Factorise: $11x - 22y^2$.

\[11x - 22y^2 = (11)(x) - (11)(2y^2)\]

\[= 11(x - 2y^2)\]

Check: $11(x - 2y^2) = 11x - 22y^2$ as required.

To extend this procedure to allow for mixed factors, we need to find the greatest common factor (GCF).

With numbers, the greatest common factor can be found by inspection. For example, the greatest common factor of 8, 24 and 36 is 4, since 4 is the largest number that divides evenly into all three numbers.

The greatest common factor of $x^3$, $x^5$ and $x^7$ is $x^3$ because $x^3$ is the highest power of $x$ that is common to all three. Note that the greatest common factor of three terms like this, must always be the lowest power.

If you study the following examples carefully, the process will become clear.

Example 5  Find the greatest common factor of $6x^7$, $4x^5$ and $10x^3$.

First consider the numbers (coefficients) in front of each term, 6, 4 and 10. The greatest common factor of these is 2, since 2 is the highest number which will divide evenly into each coefficient.

Secondly, consider the variable $x$ in each term, $x^7$, $x^5$ and $x^3$. The greatest common factor of these is $x^3$ (remember that it will always be the lowest power).

So the greatest factor of $6x^7$, $4x^5$ and $10x^3$ is $2x^3$ (the product of the factors found above).

Example 6  Factorise: $12x - 8$ by removing all common factors.

\[12x - 8 = 4(3x - 2)\]

GCF \[\text{Result of } \frac{(-8)}{4}\]

Result of \[\frac{(12x)}{4}\]

Example 7  Factorise: $4x^2 - 6x^3$ by removing all common factors.

\[4x^2 - 6x^3 = 2x^2(2 - 3x)\]

GCF \[\text{Result of } \frac{(-6x^3)}{(2x^2)}\]

Result of \[\frac{(4x^2)}{(2x^2)}\]

Example 8  Factorise: $7x^2y - 14x^2y^3 - 21x^3y$.

\[7x^2y - 14x^2y^3 - 21x^3y = 7x^2y(1 - 2y^2 - 3x)\]
Activity 1.3

Factorise:

1. $4a + 8b$
2. $2 - 6k$
3. $x^2 + xy$
4. $pq - 3q^2$
5. $x^3y - x$
6. $xyz + z^3$
7. $3x + 6x^2$
8. $5x - 15x^2y$
9. $8x^2 - 4x^3y$
10. $-15xy + 55xz$
11. $2bc - 6b^2c + 12b^3$
12. $x^6 - 4x^3$
13. $-8x^3 + 6x^2 - 14x^6$
14. $10x^3y^4 - 5x^2y^2 - 10xy^3$

1.4 Factorisation involving difference of squares

Previously we learned how to multiply two binomials. If these binomials represent the difference of two numbers $(x - k)$ and their sum $(x + k)$ of two numbers, we obtain:

$$(x - k)(x + k) = (x - k)(x) + (x - k)(k)$$

$$= (x)(x - k) + (k)(x - k)$$

$$= (x)(x) + (x)(-k) + (k)(x) + (k)(-k)$$

$$= x^2 - xk + kx - k^2$$

$$= x^2 - k^2$$

Hence this product equals the difference between the squares of the numbers, ie $(x^2 - k^2)$.

In reverse: 

$x^2 - k^2 = (x - k)(x + k)$

or 

$x^2 - k^2 = (x + k)(x - k)$

That is, take the square root of each term first, then the factors are the sum and difference of these square roots.

Example 1 Factorise: $x^2 - 16$.

Although $x^2 - 16$ has no common factors, we can see that it is in the general form of a difference of 2 squares $(x^2 - k^2)$, where $k^2 = 16$, ie $k = \pm 4$

$\therefore x^2 - 16 = (x - 4)(x + 4)$

Example 2 Factorise: $x^2 - 25$.

$x^2 - 25 = (x + 5)(x - 5)$

Example 3 Factorise: $x^2 - 9$.

$x^2 - 9 = (x + 3)(x - 3)$
Example 4  Factorise: $25 - (6x)^2$.

$$25 - (6x)^2 = (5 + 6x)(5 - 6x)$$

Example 5  Factorise: $49 - (2x)^2$.

$$49 - (2x)^2 = (7 + 2x)(7 - 2x)$$

If two processes of factorising are to be done, one of which is extracting a common factor, always deal with the common factor first.

In fact, often we can’t see the ‘difference of squares’ until the common factors are removed.

Example 6  Factorise: $2x^2 - 18$.

$$2x^2 - 18 = 2(x^2 - 9)$$
$$= 2(x^2 - 3^2)$$
$$= 2(x - 3)(x + 3)$$

Example 7  Factorise: $5 - 20p^2$.

$$5 - 20p^2 = 5(1 - 4p^2)$$
$$= 5((1)^2 - (2p)^2)$$
$$= 5(1 - 2p)(1+2p)$$

Example 8  Factorise: $81p^4 - 16$.

$$81p^4 - 16 = (9p^2 - 4)(9p^2 + 4)$$
$$= (3p - 2)(3p + 2)(9p^2 + 4)$$

Note that in this example we factorised twice using ‘difference of squares’. Not many problems are as tricky as this last one but the example illustrates that you should always examine your answer and be satisfied that you cannot factorise any further.

Activity 1.4

Factorise:

1. $x^2 - 36$
2. $m^2 - 64$
3. $y^2 - 1$
4. $100 - x^2$
5. $p^2 - q^2$
6. $(2n)^2 - 49$
7. $1 - k^2$
8. $25 - (3t)^2$
9. $7 - 28y^2$
10. $4x^2 - 16$
11. $y^2 - 1$
12. $24x^2 - 150y^2$
13. $3x^2 - 75$
14. $27ab^2 - 3ac^2$
1.5 Factorisation of trinomials

In Section 1.2, we multiplied binomials as follows:

\[(y + 7)(y + 3) = y^2 + 3y + 7y + 21 = y^2 + 10y + 21\]
\[(y - 7)(y - 3) = y^2 - 3y - 7y + 21 = y^2 - 10y + 21\]
\[(y + 7)(y - 3) = y^2 - 3y + 7y - 21 = y^2 + 4y - 21\]
and \[(y - 7)(y + 3) = y^2 + 3y - 7y - 21 = y^2 - 4y - 21\]

In this section we want to perform the reverse operation; that is, to factorise
\[y^2 + 10y + 21\] as \[(y + 7)(y + 3)\], ie the product of binomial factors.

To help us in this, we can arrive at some general patterns in factorising trinomials by studying the set of examples above.

- The first terms in each bracket are the factors of the first term in the trinomial \((y, y = y^2)\).
- The last terms in each bracket are the factors of the last term in the trinomial \((3 \times 7 = 21)\).
- The middle term of the trinomial is the sum of the last terms in each bracket \((3 + 7 = 10, -3 - 7 = -10, -3 + 7 = 4, -7 + 3 = -4)\).

So, very roughly, we are looking for two numbers which when multiplied together give the end term of our trinomial and which, added together, give the middle term of our trinomial.

That is: \[x^2 + (A + B)x + (AB) = (x + A)(x + B)\].

Example 1 Factorise: \[x^2 + 8x + 16\].

First, looking for numbers which multiply together to give 16 (the end term of our trinomial) we have as possibilities:
1, 16
2, 8
and 4, 4

However, of these, only 4 and 4 add up to 8 (which is the coefficient of the middle term in our trinomial). Therefore, 4 and 4 is the required pair.

So, \[x^2 + 8x + 16 = (x + 4)(x + 4)\].
**Example 2**  Factorise: \( x^2 + 2x - 8 \).

We want two numbers which when multiplied give \((-8)\) and which, when added, give 2 (the coefficient of the middle term). The possibilities are:

1, \(-8\)
8, \(-1\)
2, \(-4\)

and 4, \(-2\) which all multiply to give \((-8)\).

But of these, only 4 and \(-2\) add up to 2.

\[
\therefore \quad x^2 + 2x - 8 = (x + 4)(x - 2)
\]

Check: \((x + 4)(x - 2) = x(x - 2) + 4(x - 2)\)

\[
= x^2 - 2x + 4x - 8
\]

\[
= x^2 + 2x - 8
\]

The above method of trial and error may be simplified using the cross method.

**Example 3**  Factorise: \( m^2 + 4m - 12 \).

We first set up a cross as follows:

Terms for first bracket

Terms for second bracket

Factors of the first term in our trinomial

Factors of the end term

Now the factors of the first term \((m^2)\) are \(m, m\) and the possible choice of factors \((A, B)\) for the end term are:

\(-1, \quad 12\)
\(1, \quad -12\)
\(2, \quad -6\)
\(-2, \quad 6\)
\(3, \quad -4\)
\(-3, \quad 4\)

We choose the pair which, after we cross multiply, add up to the middle term of our trinomial.

So, in our case, we want:

\[(C \times B) + (D \times A) = 4m\]
So, our working will appear as:

\[
\begin{align*}
1m & + 6 \\
1m & - 2
\end{align*}
\]

since \((m-2) + (m6)\)  
\[
= -2 + 6m \\
= 4m
\]

(The dot ‘.’ indicates multiplication.)

\[
\therefore m^2 + 4m - 12 = (m + 6)(m - 2)
\]

Check:

\[
= (m + 6)(m - 2) \\
= m(m - 2) + 6(m - 2) \\
= m^2 - 2m + 6m - 12 \\
= m^2 + 4m - 12
\]

At first this method may not seem quite as simple, but it will handle all types of trinomials.

**Example 4**  
Factorise: \(2x^2 - 7x + 3\).

First notice that this time the coefficient of the first term is 2, not 1. Using the cross method:

\[
\begin{array}{c}
2x \\
1x
\end{array} \times \begin{array}{c}
-1 \\
-3
\end{array}
\]

Factors of \(2x^2\)  Factors of 3, chosen such that:  
\((2x, -3) + (1x, -1) = -6x - x\)  
\[= -7x, \text{ the middle term.}\]

\[
\therefore 2x^2 - 7x + 3 = (2x - 1)(x - 3)
\]

Check:

\[
= (2x - 1)(x - 3) \\
= 2x(x - 3) - 1(x - 3) \\
= 2x^2 - 6x - x + 3 \\
= 2x^2 - 7x + 3
\]
Example 5  Factorise: \( x^2 + 5xy - 24y^2 \).

First notice that this time the end term involves a second variable. Take a special note of how this affects our solution.

\[
\begin{align*}
1x & \quad +8y \\
1x & \quad -3y \\
\end{align*}
\]

Factors of \(-24y^2\)

\[
\therefore \quad x^2 + 5xy - 24y^2 = (x + 8y)(x - 3y)
\]

Check:

\[
\begin{align*}
(x + 8y)(x - 3y) & = x(x - 3y) + 8y(x - 3y) \\
& = x^2 - 3xy + 8xy - 24y^2 \\
& = x^2 + 5xy - 24y^2
\end{align*}
\]

Example 6  Factorise: \( 5x^2 - 26x + 5 \).

\[
\begin{align*}
5x & \quad -1 \\
1x & \quad -5 \\
\end{align*}
\]

\[
\therefore \quad 5x^2 - 26x + 5 = (5x - 1)(x - 5)
\]

Check:

\[
\begin{align*}
(5x - 1)(x - 5) & = 5x(x - 5) - 1(x - 5) \\
& = 5x^2 - 25x - x + 5 \\
& = 5x^2 - 26x + 5
\end{align*}
\]

Activity 1.5

Factorise:

1. \( x^2 + 3x + 2 \)
2. \( x^2 - x - 2 \)
3. \( x^2 - 7x + 12 \)
4. \( x^2 + 2x - 15 \)
5. \( x^2 + 5x - 24 \)
6. \( 2x^2 + 9x + 7 \)
7. \( 3x^2 + 11x - 4 \)
8. \( 5x^2 - 13x - 6 \)
1.6 Algebraic fractions

In dealing with algebraic fractions, we frequently need to factorise. Let us first look at multiplication and division.

We cancel arithmetic fractions by factoring the numerator and denominator.

\[
\frac{12}{84} = \frac{(2)(2)(3)}{(2)(2)(3)(7)} = \frac{2 \times 3 \times 1}{2 \times 3 \times 7} = \frac{1 \times 1 \times 1}{7} = \frac{1}{7}
\]

‘Cancelling’ is effected by pairing up a factor in the numerator with an equal factor in the denominator and using the properties that \( \frac{a}{a} = 1 \), and \( k \times 1 = k \).

To simplify algebraic fractions we can proceed in exactly the same way.

Example 1

Simplify: \( \frac{x^2 - x - 6}{x^2 - 7x + 12} \)

First we completely factor the numerator and denominator.

\[
\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)}
\]

Pairing the common factors in the numerator and denominator and cancelling:

\[
\frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} = \frac{1 \times (x + 2)}{(x - 4)}
\]

Using the property that \( k \times 1 = k \)

\[
1 \times \frac{x + 2}{x - 4} = \frac{x + 2}{x - 4}
\]

Example 2

Simplify: \( \frac{2x^2 - 3x + 1}{4x^2 - 1} \)

\[
\frac{2x^2 - 3x + 1}{4x^2 - 1} = \frac{(2x - 1)(x - 1)}{(2x - 1)(2x + 1)} = \frac{x - 1}{2x + 1}
\]

Because we cancel using factors, the simplification of products and quotients of rational algebraic expressions follows along similar lines.
**Example 3**  
Simplify: \[ \frac{(5t + 5) \times t^2 - 4t + 4}{t - 2} \times \frac{t^2 - 1}{t^2 - 4t + 4} \]

Factor: \[ \frac{(5t + 5) \times t^2 - 4t + 4}{t - 2} \times \frac{t^2 - 1}{t^2 - 4t + 4} = \frac{5(t + 1)}{t - 2} \times \frac{(t - 1)(t + 1)}{(t - 1)(t + 1)} \]

Pair like factors in the numerator and denominator:

\[ \frac{5(t + 1)(t - 2)(t - 2)}{(t + 1)(t - 2)(t - 1)} = \frac{5 	imes t + 1}{t - 2} \times \frac{t - 2}{t - 1} \times \frac{t - 2}{t - 1} \]

Use \( \frac{a}{a} = 1 \) and \( k \times 1 = k \) to obtain:

\[ \frac{5}{1} \times 1 \times \frac{t - 2}{t - 1} = \frac{5(t - 2)}{t - 1} \]

Quotients are found by using the definition of division: \( x \div \frac{a}{b} = x \times \frac{b}{a} \) and then proceeding as before.

**Example 4**  
Simplify: \[ \frac{p^2 + 3p}{p^2 + 2p - 3} \div \frac{p}{p + 1} \]

Use the definition of division:

\[ \frac{p^2 + 3p}{p^2 + 2p - 3} \div \frac{p}{p + 1} = \frac{p^2 + 3p}{p^2 + 2p - 3} \times \frac{p + 1}{p} \]

Now proceed as before:

\[ \frac{p^2 + 3p}{p^2 + 2p - 3} \times \frac{p + 1}{p} \]

\[ = \frac{p(p + 3)}{(p + 3)(p - 1)} \times \frac{p + 1}{p} \]

\[ = \frac{p \times p + 3 \times p + 1}{p \times p + 3 \times p - 1} \]

\[ = 1 \times 1 \times \frac{p + 1}{p - 1} \]

\[ = \frac{p + 1}{p - 1} \]
Algebraic fractions are added using exactly the same algorithm as for arithmetic fractions:

\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}
\]

\[
\frac{p}{q} = \frac{p \times 1}{q} = \frac{p \times a}{q} = \frac{pa}{qa}
\]

We obtain the fractions to be added or subtracted in an equivalent form with common denominators, by using the property:

\[
\frac{p}{q} = \frac{p \times a}{q} = \frac{pa}{qa}
\]

When adding or subtracting algebraic fractions, it is best to choose the common denominator to be the product of all the denominators. Then it is easy to see which factor to multiply both numerator and denominator by!

**Example 1**  
Simplify: \(\frac{2}{x} + \frac{1}{x-1}\)

Choose the common denominator to be the product of the denominators: \(x(x - 1)\).

(There is no need to multiply this out.)

To obtain the first fraction with this as a denominator, we must multiply top and bottom by \(x - 1\). The second fraction is obtained with the new denominator by multiplying top and bottom by \(x\).

So:

\[
\frac{2}{x} + \frac{1}{x-1} = \frac{2(x-1) + 1 \cdot x}{x(x - 1)}
\]

\[
= \frac{2(x-1) + x}{x(x - 1)}
\]

\[
= \frac{2x - 2 + x}{x(x - 1)}
\]

\[
= \frac{3x - 2}{x(x - 1)}
\]

**Example 2**  
Simplify: \(\frac{x-1}{x-2} - \frac{x+1}{x+2}\)

Obtain the fractions with a common denominator:

\[
\frac{(x-1)(x+2)}{(x-2)(x+2)} - \frac{(x+1)(x-2)}{(x+2)(x-2)}
\]
Now use the common denominator and multiply out the numerator to make it easier to subtract:

\[
\frac{x^2 + x - 2 - (x^2 - x - 2)}{(x - 2)(x + 2)}
\]

\[
= \frac{x^2 + x - 2 - x^2 + x + 2}{x^2 - 4}
\]

\[
= \frac{2x}{x^2 - 4}
\]

Notice that sometimes the denominator can be multiplied out to obtain a simpler final result.

In some cases, the denominators should be factorised first in order to determine the desired common denominator as in the example below.

**Example 3**  Simplify: \( \frac{1}{t^2 - 1} + \frac{t}{t^2 + 2t + 1} \)

\[
\frac{1}{t^2 - 1} + \frac{t}{t^2 + 2t + 1} = \frac{1}{(t - 1)(t + 1)} + \frac{t}{(t + 1)(t + 1)}
\]

\[
= \frac{t + 1}{(t - 1)(t + 1)(t + 1)} + \frac{t(t - 1)}{(t + 1)(t + 1)(t - 1)}
\]

\[
= \frac{t + 1 + t^2 - t}{(t - 1)(t + 1)^2}
\]

\[
= \frac{t^2 + 1}{(t - 1)(t + 1)^2}
\]

Note that in this last example the common denominator wasn’t the product of the two original denominators. It was easier to obtain the common denominator by working out the simplest string of factors that needed to be used in order that the denominators are the same and then multiplying top and bottom on each fraction so as to get that denominator. This happens when there are already some common factors in the denominators.
Activity 1.6

Simplify:

1. \( \frac{x^2 - 4}{x^2 - 2x} \)

2. \( \frac{3t^2 - 27t + 24}{2t^3 - 16t^2 + 14t} \)

3. \( \frac{x^2 - y^2}{x + y} \times \frac{x^2 + 2xy + y^2}{y - x} \)

4. \( \frac{p^2 - 4}{p^2 + 2p} \times \frac{p^2}{p - 2} \)

5. \( \frac{2a - 2}{a^2 - 2a - 8} \div \frac{a^2 - 1}{a^2 + 5a + 4} \)

6. \( \frac{z^2 + 2z}{3z^2 - 18z + 24} + \frac{z^2 - z - 6}{z^2 - 4z + 4} \)

7. \( \frac{1}{x - 1} + \frac{1}{x + 1} \)

8. \( \frac{2}{t} - \frac{1}{t + 1} \)

9. \( \frac{p}{p + 2} + \frac{1}{p + 1} \)

10. \( \frac{a + 1}{a - 1} - \frac{a - 1}{a + 1} \)

11. \( \frac{1}{x^2 - x - 2} + \frac{1}{x^2 - 1} \)

12. \( 1 - \frac{t}{t - 1} \)

1.7 Solving quadratic equations by factoring

A quadratic equation is an equation in which the unknown is squared.

The general format is \( ax^2 + bx + c = 0 \).

Because of the term \( x^2 \), quadratic equations generally have two solutions, but sometimes there is only one. It is also possible that there is no solution such as in the quadratic equation \( x^2 + 4 = 0 \) because here we require that \( x^2 = -4 \). This is, of course, impossible because no numbers squared (even a negative number) will give us a negative result.

There are various methods for solving quadratic equations. In this section we look at the factoring method. Because, as we observed earlier, not all polynomials can be factored, this method has some limitations.

Let us start with an example.

Example 1  Solve by factoring: \( x^2 + 5x + 4 = 0 \).

We can factorise the trinomial to get:

\( x^2 + 5x + 4 = (x + 4)(x + 1) \)

Now \( x^2 + 5x + 4 = 0 \)

So \( (x + 4)(x + 1) = 0 \)

If two quantities multiplied together give a result of zero, one or the other (or both) of the quantities must be zero.
Here we have the quantities \((x + 4)\) and \((x + 1)\) multiplying together to give zero.

So we can say either \(x + 4 = 0\) or \(x + 1 = 0\)

Now we have two first degree equations which we can solve by the methods of Section 2.

\[ x = -4 \quad \text{or} \quad x = -1 \]

Check that these values do make the original statement true.

If \(x = -4\),

\[ x^2 + 5x + 4 = (-4)^2 + 5(-4) + 4 = 16 - 20 + 4 = 0 \]

If \(x = -1\),

\[ x^2 + 5x + 4 = (-1)^2 + 5(-1) + 4 = 1 - 5 + 4 = 0 \]

So these values of \(x\) do make the equation true.

The solutions are \(x = -4\) or \(x = -1\).

**Example 2** Solve by factoring: \(x^2 - 8x + 12 = 0\).

\[ x^2 - 8x + 12 = 0 \]
\[ (x - 6)(x - 2) = 0 \]

\[ x - 6 = 0 \quad \text{or} \quad x - 2 = 0 \]
\[ x = 6 \quad \text{or} \quad x = 2 \]

**Example 3** Solve by factoring: \(x^2 + 5x - 6 = 0\).

\[ x^2 + 5x - 6 = 0 \]
\[ (x + 6)(x - 1) = 0 \]

\[ x + 6 = 0 \quad \text{or} \quad x - 1 = 0 \]
\[ x = -6 \quad \text{or} \quad x = 1 \]

When the right hand side of the equation is not zero, it must be rearranged before factoring.
Example 4  Solve by factoring: \( x^2 + x - 2 = 10 \).

\[ x^2 + x = 12 \]

Rearrange:

\[ x^2 + x - 12 = 0 \]

Factor:

\[ (x + 4)(x - 3) = 0 \]

\[ x = -4 \] or \[ x = 3 \]

Example 5  Solve by factoring: \((y + 3)^2 = 16\).

\[ (y + 3)^2 = 16 \]

Multiply out:

\[ y^2 + 6y + 9 = 16 \]

Rearrange:

\[ y^2 + 6y - 7 = 0 \]

Factor:

\[ (y + 7)(y - 1) = 0 \]

\[ y = -7 \] or \[ y = 1 \]

Example 6  Solve by factoring: \( x^2 = 5x \).

\[ x^2 = 5x \]

Rearrange:

\[ x^2 - 5x = 0 \]

This is not a trinomial, but we can extract a factor:

\[ x(x - 5) = 0 \]

Again, we have two quantities multiplying together to give zero. So we can say:

\[ x = 0 \] or \[ x - 5 = 0 \]

\[ x = 0 \] or \[ x = 5 \]

Example 7  Solve by factoring: \((y + 2)(2y - 3) = 6(y - 1)\).

This is not in the form we require, so we multiply the terms out and collect them.

\[ (y + 2)(2y - 3) = 6(y - 1) \]

Multiply out:

\[ 2y^2 - 3y + 4y - 6 = 6y - 6 \]

\[ 2y^2 + y - 6 - 6y + 6 = 0 \]

Collect terms:

\[ 2y^2 - 5y = 0 \]

Factor:

\[ y(2y - 5) = 0 \]
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Example 8  Solve by factoring: \( x^2 = 9 \).
Again we need to rearrange to give zero on the right hand side.
\( x^2 = 9 \) gives \( x^2 - 9 = 0 \)
This is a difference of squares, so again we can use factorisation:
\[
(x - 3)(x + 3) = 0
\]
x - 3 = 0 or x + 3 = 0
x = 3 or y = (-3)
This method avoids the trap of looking at the problem and deciding by inspection that the solution is \( x = 3 \). You can see that \( x = -3 \) is also a solution.

Example 9  Solve by factoring: \( \frac{x^2}{2} = 8 \).
Multiply both sides by 2: \( x^2 = 16 \)
x^2 - 16 = 0
(x - 4)(x + 4) = 0
x - 4 = 0 or x + 4 = 0
x = 4 or y = (-4)

Activity 1.7
Solve by factoring:
1. \( x^2 - 2x + 1 = 0 \)  
2. \( x^2 + 2x - 15 = 0 \)
3. \( x^2 - 14x = 0 \)  
4. \( x^2 = 3x + 28 \)
5. \( p^2 = 100 \)  
6. \( (k - 3)^2 = 4 \)
7. \( x^2 + 3x + 2 = 12 \)  
8. \( 5c^2 = 12c \)
9. \( 7x^2 = 28 \)  
10. \( 9y^2 = 1 \)
11. \( x^2 - 7x = 0 \)  
12. \( (v + 4)(v + 2) = 8(1 + 3v) \)
1.8 Solving quadratic equations by formula

Another method for solving quadratic equations is by a process called completing the square. We shall not study this process but mention it because if this process is employed on the quadratic equation in its most general form:

\[ ax^2 + bx + c = 0 \]

then we obtain the following general solutions to this equation.

\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

These two solutions are written together using the ± sign. The result is called the quadratic formula for finding the solution to quadratic equations.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Particular quadratic equations can be solved by substituting values for \( a, b, c \).

**Example 1** Use the formula to solve \( 3p^2 - 4p - 2 = 0 \).

Give your answers correct to two decimal places.

First determine the values for \( a, b, c \). \( a = 3, \ b = -4, \ c = -2 \)

Now substitute into the formula. In this example the variable name is \( p \) so:

\[ p = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)} \]

Hence \( p = \frac{4 \pm \sqrt{16 + 24}}{6} = \frac{4 \pm \sqrt{40}}{6} \)

Then \( p = \frac{4 + \sqrt{40}}{6} \) or \( p = \frac{4 - \sqrt{40}}{6} \)

\( p = 1.72 \) or \( p = (-0.39) \)

**Activity 1.8**

Use the formula to solve these quadratic equations. Express the results correct to two decimal places.

1. \( 2x^2 - 5x - 12 = 0 \)
2. \( x^2 - 3x = -2 \)
3. \( 4t^2 - t - 1 = 0 \)
4. \( 5p + 2 = 12p^2 \)
5. \( b^2 + 3b + 1 = 0 \)
6. \( k^2 - 14k + 12 = 0 \)
1.9 Quadratic equation word problems

As with all word problems you should carefully define the variable you wish to solve for by a statement such as 'let x be ….' Then write down any other quantities in terms of x, and look for a statement in the question that relates all the quantities together.

Drawing a diagram is generally very useful.

Always give your answer in expanded written form, using correct units.

Example 1  The perimeter of a rectangular field is 500 metres, and its area is 14 400 square metres. Find the length of the sides.

Let the length of one of the sides be $L$ metres.

The length of the other side is $250 - L$ metres, as shown in the following diagram.

$$\begin{align*}
&\quad 250 - L \\
&\quad L \quad L \\
&\quad 250 - L \\
&\text{Half perimeter = 250 m}
\end{align*}$$

$$Area = L(250 - L)$$

Now translate 'its area is 14 400' into an equation:

$L(250 - L) = 14 400$

ie $250L - L^2 = 14 400$

Transposing, we have:

$L^2 - 250L + 14 400 = 0$

$(L - 90)(L - 160) = 0$, hence $(L = 90 \text{ m or } 160 \text{ m})$

(Do not put the unit ‘metres’, in the equation.)

The other side is $250 - 90 = 160 \text{ m}$ or $250 - 160 = 90 \text{ m}$.

Therefore there is only one set of lengths in the final solution.

The length of the sides is 90 m and 160 m.
**Example 2**  
A man travelled from ‘A’ to ‘B’, 108 km from ‘A’. He then travelled from ‘B’ to ‘A’ in 4½ hours less at a speed of 2 km per hour faster. At what speed did he travel from ‘A’ to ‘B’?

Let the required speed be \( v \) (km per hour).

Then the time required to travel from ‘A’ to ‘B’ is \( \frac{108}{v} \) hours.

From ‘B’ to ‘A’, the speed is \( v + 2 \) (km per hour).

The time required = \( \frac{108}{v + 2} \)

Now, we can translate ‘... 4½ (\( \frac{9}{2} \)) hours less’ to an equation:

\[
\frac{108}{v + 2} - \frac{108}{v} = \frac{9}{2}
\]

Multiplying each term by \( 2v(v + 2) \) we have:

\[
108 \times 2 = 108 \times 2(v + 2) - 9v(v + 2)
\]

ie \( 0 = -216v + 216v + 432 - 9v^2 - 18v \)

ie \( 9v^2 + 18v - 432 = 0 \)

\[
\frac{9v^2 + 18v - 432}{9} = 0
\]

\( (v - 6)(v + 8) = 0 \), hence

\( v = 6 \) or \( v = -8 \) (discard)

Final solution: The required speed is 6 km per hour.

**Activity 1.9**

1. If a train travelled 5 kilometres an hour faster it would take one hour less to travel 210 kilometres. What time does it take?

2. The perimeter of one square exceeds that of another by 100 metres, and the area of the larger square exceeds three times the area of the smaller by 325 square metres. Find the length of their sides.

3. A lawn 50 metres long and 34 metres wide has a path of uniform width round it. If the area of the path is 540 square metres, find its width.

4. A hall can be paved with 200 square tiles of a certain size; if each tile were one centimetre longer each way it would take 128 tiles. Find the length of each tile.

5. Two rectangles contain the same area, 480 square metres. The difference of their lengths is 10 metres, and of their breadths 4 metres. Find the dimensions of each rectangle.

6. ‘A’ and ‘B’ are two stations 300 kilometres apart. Two trains start simultaneously from ‘A’ and ‘B’, each to the opposite station. The train from ‘A’ reaches ‘B’ in nine hours; the train from ‘B’ reaches ‘A’ four hours after they meet. Find the rate at which each train travels.
Assessment 1

1. Simplify: \(2x(1-x) - 3(2 - x^2)\)

2. Multiply out: \((2x - 3)(x + 5)\)

3. Factorise: \(3t^2 - 6t\)

4. Factorise: \(25 - 16p^2\)

5. Factorise: \(x^2 - 6x + 9\)

6. Factorise: \(4z^2 + 4z - 15\)

7. Use the quadratic formula to solve the quadratic equation: \(3t^2 - 2t - 2 = 0\). Express the solutions correct to three decimal places.

8. Simplify: \(\frac{x^2 - 4}{x^2 + 2x - 3} \div \frac{x^2 - x - 6}{x^2 - 9}\)

9. Simplify: \(\frac{1}{x} - \frac{x}{1-x}\)

10. A rectangular plot, 4 m by 8 m, is to be used for a garden. It is decided to put a pavement inside the entire border so that 12 m² of the plot is left for flowers. How wide should the pavement be?

11. A rectangular garden is 2 metres longer than it is wide. If the width is doubled and the length diminished by 4 metres, the area is unchanged. What are the original dimensions?
Answers to activities

Activity 1.1
1. $b^2 - b - 10$
2. $p^3 - p^2 + 12$
3. $x^3y - 5x^2y^2 + xy^3$
4. $a^2 + b^2 - 3a - 2b$

Activity 1.2
1. $xy + xb + ay + ab$
2. $8xy + 6x - 4y - 3$
3. $6p^2 - 17pq + 5q^2$
4. $5x^3 - x^2 + 30x - 6$
5. $12 + 3y - 4xy - xy^2$
6. $ax^3 - x^2z^2 - axyz + yz^3$
7. $y^2 + 6y + 8$
8. $x^2 - x - 30$
9. $m^2 - 6m + 5$
10. $x^2 + 4x - 21$
11. $2x^2 - 13x + 20$
12. $5x^2 + 22xy + 8y^2$
13. $56p^2 - 13pq - 3q^2$
14. $r^4 + r^2 - 2$
15. $x^2 - 4x + 4$
16. $4m^2 - 12m + 9$
17. $36 - 12y + y^2$
18. $4m^2 - 12mn + 9n^2$
19. $x^2 - 4xy + 4y^2$
20. $m^2 - 6mn + 9n^2$

Activity 1.3
1. $4(a + 2b)$
2. $2(1 - 3k)$
3. $x(x + y)$
4. $q(p - 3q)$
5. $x(xy - 1)$
6. $z(xy + z^2)$
7. $3x(1 + 2x)$
8. $5x(1 - 3xy)$
9. $4x^2(2 - xy)$
10. $-5x(3y - 11z)$
11. $2b(c - 3bc + 6b^2)$
12. $x^3(x^3 - 4 + x^2)$
13. $-2x^2(4x - 3 + 7x^3)$
14. $5xy^2(2x^2y^2 - x - 2y)$

Activity 1.4
1. $(x + 6)(x - 6)$
2. $(m + 8)(m - 8)$
3. $(y + 1)(y - 1)$
4. $(10 + x)(10 - x)$
5. $(p + q)(p - q)$
6. $(2n + 7)(2n - 7)$
7. $(1 + k)(1 - k)$
8. $(5 + 3t)(5 - 3t)$
9. $7(1 + 2y)(1 - 2y)$
10. $4(x + 2)(x - 2y)$
11. $y(1 + y)(1 - y)$
12. $6(2x + 5y)(2x - 5y)$
13. $3(x + 5)(x - 5)$
14. $3a(3b + c)(3b - c)$
Activity 1.5
1. \((x + 1)(x + 2)\)  
2. \((x + 1)(x - 2)\)  
3. \((x - 3)(x - 4)\)  
4. \((x + 5)(x - 3)\)  
5. \((x + 8)(x - 3)\)  
6. \((x + 1)(2x + 7)\)  
7. \((3x - 1)(x + 4)\)  
8. \((5x + 2)(x - 3)\)

Activity 1.6
1. \(\frac{x + 2}{x}\)  
2. \(\frac{3(t - 8)}{2t(t - 7)}\)  
3. \(- (x + y)^2\)  
4. \(p\)  
5. \(\frac{2(a + 4)}{(a - 4)(a + 2)}\)  
6. \(\frac{z(z - 2)}{3(z - 4)(z - 3)}\)  
7. \(\frac{2x}{x^2 - 1}\)  
8. \(\frac{t + 2}{t(t + 1)}\)  
9. \(\frac{p^2 + 2p + 2}{(p + 2)(p + 1)}\)  
10. \(\frac{4a}{a^2 - 1}\)  
11. \(\frac{2x - 3}{(x - 2)(x + 1)(x - 1)}\)  
12. \(\frac{1}{1-t}\)

Activity 1.7
1. \(x = 1\)  
2. \(x = 3\) or \((-5)\)  
3. \(x = 0\) or \(14\)  
4. \(x = 7\) or \((-4)\)  
5. \(p = 10\) or \((-10)\)  
6. \(k = 1\) or \(5\)  
7. \(x = 2\) or \((-5)\)  
8. \(c = 2.4\) or \(0\)  
9. \(x = 2\) or \((-2)\)  
10. \(y = \frac{1}{3}\)  
11. \(x = 0\) or \(x = 7\)  
12. \(v = 0\) or \(18\)

Activity 1.8
1. \(x = 4\) or \((-1.5)\)  
2. \(x = 2\) or \(1\)  
3. \(t = 0.64\) or \((-0.39)\)  
4. \(p = 0.67\) or \((-0.25)\)  
5. \(b = (-0.38)\) or \((-2.62)\)  
6. \(k = 13.08\) or \(0.92\)

Activity 1.9
1. 7 hours  
2. 55 metres, 30 metres  
3. 3 metres  
4. 4 cm, 5 cm  
5. 30 m by 16 m and 40 m by 12 m  
6. 33.3 km/h and 36.0 km/h (1 dp)
Section 2 – Functions

2.1 Definition of a relation

You have probably used a street directory, which consists of a list of street names and a series of maps. To find, for example, Boon Court in Marmion, you note the map number and an ordered pair such as (7, D) which locates the street on the map with reference to a rectangular grid which has numbers on one axis and letters on the other.

A set of such ordered pairs is called a relation. If, as is usually done, we call the set by a capital letter (B may be appropriate here) and use the curly bracket {...} notation for sets, then formally we would write for the Boon Court location \( B = \{(7, D)\} \).

In our example, the relation has only one ordered pair or element. Normally, we have many elements in the set such as in set \( X \):

\[
X = \{(1, A), (2, B), (3, C), (4, C), (1, D)\}.
\]

Also, in mathematics we generally deal with ordered pairs that are of the form \((x, y)\) in which \(x\) designates the position on the horizontal axis and \(y\) designates the position on the vertical axis. A typical example would be:

\[
S = \{(0, 0), (2, 1), (2, -1), (3, 3), (-1, 4), (5, 5)\}.
\]

This relation can easily be graphed as is done below:

**Example 1** Graph the relation \( S \).

There are two important concepts associated with relations. The first concept is called the domain of a relation, which is the set of all first members of the ordered pairs, usually the \(x\) values. The range is the set of all second members of the ordered pairs, usually the \(y\) values.
Example 2  Write down the domain and range of sets B, X and S listed below.

<table>
<thead>
<tr>
<th>Set</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>{7}</td>
<td>{D}</td>
</tr>
<tr>
<td>X</td>
<td>{1, 2, 3, 4}</td>
<td>{A, B, C, D}</td>
</tr>
<tr>
<td>S</td>
<td>{-1, 0, 2, 3, 5}</td>
<td>{-1, 0, 1, 3, 4, 5}</td>
</tr>
</tbody>
</table>

Often, a formula or algebraic rule is used to describe the relationship between the first and second members of a relation.

For example, the rule may be: \( y = x^2 + 3 \).

Calling this relation \( C \), to incorporate the fact that a relation is a set, we formally write:

\[ C = \{(x, y) \mid y = x^2 + 3\} \]

where \( \mid \) is read as ‘such that’.

By writing a relation in this format it is clear that the value of \( y \) is obtained from the value of \( x \) by applying the stated rule. For this reason we call \( x \) the **independent variable** and \( y \) the **dependent variable**.

Example 3  If \( Q = \{(x, y) \mid y = -x + 3\} \) and the domain of \( Q \) is \{0, 1, 2, 3, 4, 5\} list the relation.

By substituting the domain values into the formula, the corresponding values in the range are produced and the relation can be listed. A table is useful for this purpose.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
</tr>
</tbody>
</table>

Hence \( Q = \{(0, 3), (1, 2), (2, 1), (3, 0), (4, -1)\} \)

There is no requirement for a relation to have an equal sign in the formula. Symbols such as > (greater than), < (less than), \( \geq \) (greater than or equal to) and \( \leq \) (less than or equal to) can be used. The relations are then generally called **inequalities**.

Example 4  If the numbers in the domain and range of \( Z = \{(x, y) \mid y > x + 2\} \) must belong to the set \{-2, -1, 0, 1, 2\}, list the ordered pairs in the relation.

Starting with \(-2\), the ordered pairs \((-2, 1)\) and \((-2, 2)\) are possible. If \( x = -1 \), the ordered pair \((-1, 2)\) is possible. No other ordered pairs can be generated.

Hence \( Z = \{(-2, 1), (-2, 2), (-1, 2)\} \)

All relations discussed so far are examples of so-called **finite** relations. In these relations we can actually list all the ordered pairs and, for that matter, the values in the domain and range.
It should be clear that to describe an infinite relation we must use the formula method. To designate the domain and range values of such a relation we again use symbols such as >.

Also the symbol \( \mathbb{R} \) is frequently used to indicate the set of all Real Numbers.

Consider this example.

**Example 5** Let \( L = \{(x, y) \mid y = 2x + 1, x > 0\} \). What is the range?

Here the domain is \( \{x \mid x \geq 0\} \). This means that the minimum domain value is 0.

Substituting in \( y = 2x + 1 \) gives \( y = 2(0) + 1 = 1 \). Thus the minimum \( y \) value is 1 and hence the range is \( \{y \mid y \geq 1\} \).

The domain and range of a relation can generally be easily identified from a graph.

**Example 6** What is the domain and range of the relation described by the graph?

In this example, there is no restriction on the independent variable but the dependent variable will always be non-negative. Hence the domain is \( \mathbb{R} \) while the range is \( \{y \mid y \geq 0\} \) which is sometimes written as \( \mathbb{R} \geq 0 \).

**Activity 2.1**

1. Complete the following sentences by supplying the missing words:
   a) \( (x, y) \) is called an \( .............. \) pair.
   b) \( x \) is called the first \( .............. \).
   c) The set of first elements is called the \( .............. \).
   d) The set of second elements is called the \( .............. \).
2. A relation $Q$ is defined as a set of ordered pairs $(x, y)$ such that $y = x - 2$ with domain $\{0, 1, 2, 3, 4\}$. List the range of $Q$.

3. If the domain and range values belong to the set $\{0, 1, 2\}$, list the ordered pairs in the following relations.
   a) $S = \{(x, y) \mid y < 2x\}$
   b) $T = \{(x, y) \mid y > x - 2\}$
   c) $U = \{(x, y) \mid y < x\}$
   d) $V = \{(x, y) \mid x \leq 2\}$

4. State the domain and range of the relations.
   a) $S = \{(x, y) \mid y = 2x\}$
   b) $T = \{(x, y) \mid x = 2\}$
   c) $U = \{(x, y) \mid y = \sqrt{x}\}$
   d) $V = \{(x, y) \mid y = \frac{1}{x}\}$

5. Give the domain and range of the circle relation defined by $\{(x, y) \mid x^2 + y^2 = 4\}$.

2.2 Definition of a function

Consider a situation in which we want to order the number of sausages on the basis of the number of people who are likely to attend a barbecue. This information could be conveyed by means of a relation which in part might read $\{... (65, 116), (66, 116), (67, 120), (68, 120), (69, 124) ...\}$.

Naturally, we want one indication for when 66 people attend, one indication for when 67 attend, and so on. In other words, we cannot have an ordered pair in our relation that has the same first element with two or more different second members.

Indeed, a relation such as $\{... (65, 116), (66, 116), (66, 120), (69, 125), (69, 134) ...\}$ would be nonsensical in this situation because we would not know how many sausages to order if 66 people are likely to attend.
There are many situations when it is desirable for relations to have all their ordered pairs with first members that have unique second members. Such relations are called functions. Let us look at some examples.

**Example 1** Which of these relations are functions?

a) \( A = \{(1, 2), (2, 3), (3, 4)\} \)

b) \( B = \{(5, 1), (5, 2), (4, 3)\} \)

c) \( C = \{(1, 1)\} \)

d) \( D = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\} \)

Relations A, C and D are functions, but B is not, because 5 is associated with, or to use the mathematical terminology, 'maps' to both 1 and 2 which is not allowed for functions. Note that a function can consist of a single ordered pair like function C. Also note that different domain values are allowed to be mapped to the same range value such as in relation D.

In the above example, function A and C are examples of so-called 'one-to-one' functions while function D is an example of a 'many-to-one' function. It is obviously impossible to have a 'one-to-many' or 'many-to-many' function.

When a relation is stated in terms of a formula, it is not always immediately obvious if the relation is a function. If the relation is finite, it is best to 'generate' all ordered pairs that belong to the relation. If, for a particular first member (usually \( x \)), more than one second members (usually \( y \)) can be found, the relation is not a function.

**Example 2** Confirm that the relation \( G = \{(x, y) | y^2 = x, x = 0 \text{ or } 1\} \) is not a function.

Generating the ordered pairs using a table we obtain:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>−1</td>
</tr>
</tbody>
</table>

Hence \( G = \{(0, 0), (1, 1), (1, −1)\} \) which confirms that G is not a function.

**Example 3** Is \( H = \{(x, y) | y = 2x + 1, x \text{ an integer}\} \) a function?

Generating typical ordered pairs produces the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>−8</td>
</tr>
<tr>
<td>−1</td>
<td>−1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Hence \( H = \{\ldots (-2, -3), (-1, -1), (0, 1), (1, 3), (2, 5)\ldots \} \).
No two ordered pairs have the same first member and hence the relation is likely to be a function.

A better method for determining whether a relation is a function is by drawing a graph and applying a very simple test, the **vertical line test**. Because no ordered pair in a function can have a different second element for the same first element, any vertical line drawn must not intersect the graph of a function in more than one point.

**Example 4**  
Show that the relation drawn is not a function.

The vertical line through $x = 1$, will pass through two points, ie $(1, 1)$ and $(1, 2)$. Consequently, the relation drawn is not a function.
The vertical line test is especially useful for when the relation is infinite.

**Example 5** Which of the following represent graphs of functions?

- **a)**
  ![Graph a](image)

- **b)**
  ![Graph b](image)

- **c)**
  ![Graph c](image)

- **d)**
  ![Graph d](image)

- **e)**
  ![Graph e](image)

- **f)**
  ![Graph f](image)

In b) and c) a vertical line can be drawn that will intersect the graph in two points. This is not possible in the other graphs. Hence a), d), e) and f) represent functions.

Later in this section we will investigate some of the functions drawn in the last example. For the moment we give their names: a) is a quadratic function, d) is a linear function, e) is an exponential function and f) is a trigonometric function.
Activity 2.2

1. Determine whether any of the following relations are functions.
   a) \{(1, 0), (2, 0), (3, 0), (4, 0)\}  
   b) \{(0, 1), (0, 2), (0, 3), (0, 4)\}  
   c) \{(1, 1), (2, 2), (3, 3), (4, 4)\}  
   d) \{(0, 1), (1, 0), (2, 3), (3, 2)\}

2. Let the domain of the following relations be \{0, 1, 2\}. Which are functions?
   a) \(P = \{(x, y) \mid y = x\}\)  
   b) \(Q = \{(x, y) \mid y = x^2\}\)  
   c) \(R = \{(x, y) \mid y^2 = x\}\)  
   d) \(S = \{(x, y) \mid y = 3x\}\)

3. Write T (true) or F (false).
   a) All functions are relations.  
   b) All relations are functions.  
   c) The graph of a function is always a line.  
   d) A single ordered pair is a function.

4. Determine which represent graphs of functions.
   a)  
   b)  
   c)  
   d)
2.3 Function notation

There are various ways of writing a function. Some of these we have already encountered when we discussed relations. Because functions are special relations we can use the ordered pair notation or the formula notation which we used for relations. There is, however, one special type of notation that is almost exclusively used for functions when a lower case letter, generally f, is used to designate a function. It is the ‘f of x’ notation, commonly referred to as function notation.

In function notation a relation such as \( f = \{(x, y) \mid y = 2x + 1\} \) is written as \( f(x) = 2x + 1 \). Note that \( f(x) \) is actually the second member of the ordered pair for a given first member \( x \). Thus by applying \( f \), ordered pairs of the form \((x, f(x))\) are generated. The notation clearly shows why the variable \( x \) is called the independent variable and the variable \( f(x) \) is called the dependent variable since \( f(x) \) depends on \( x \).

The notation is especially useful because tables do not need to be set up to generate ordered pairs. Further, rather than writing that for example ‘\( y = 3 \) for \( x = 1 \)’, we can now simply write \( f(x) = 3 \). We can now also ask questions such as ‘Calculate \( f(-1) \)’ rather than asking ‘Calculate \( y \) when \( x \) equals \((-1)\)’. You will find many applications of this notation if you ever study a higher level mathematics subject such as calculus where you may encounter expressions such as \( f(x) \).

Example 1  
If \( f(x) = 2x \), calculate \( f(1) \), \( f(-1) \), \( f(z) \) and \( f(x - 2) \).

In this function the independent variable is multiplied by two to produce the dependent variable \( f(x) \). In particular:

\[
\begin{align*}
  f(1) &= 2(1) = 2 \\
  f(-1) &= 2(-1) = -2 \\
  f(z) &= 2(z) = 2z \\
  f(x - 2) &= 2(x - 2) = 2x - 4
\end{align*}
\]

Example 2  
Let \( g(x) = \sqrt{x-2} \). Calculate if possible \( g(102) \), \( g(6) \), \( g(2) \) and \( g(0) \).

\[
\begin{align*}
  g(102) &= \sqrt{102-2} = \sqrt{100} = 10 \\
  g(6) &= \sqrt{6-2} = \sqrt{4} = 2 \\
  g(2) &= \sqrt{2-2} = \sqrt{0} = 0 \\
  g(0) &= \sqrt{0-2} = \sqrt{-2} 
  \text{not a real solution}
\end{align*}
\]

In Example 2 we were unable to calculate \( g(0) \). In fact for \( g \), the domain is \( \mathbb{R} \geq 2 \) while the range is \( \mathbb{R} \geq 0 \).

Another advantage of function notation is that functions can be combined to produce a composite function. We use the notation \( f(g(x)) \), meaning that \( f \) is applied on the value of \( g(x) \). Thus with this composite function notation, we work from the inside out.
Consider this example.

**Example 3**  Let \( f(x) = 2x + 1 \) and \( g(x) = x^2 \).

Calculate \( f(g(x)) \), \( g(f(x)) \), \( f(f(x)) \) and \( g(g(x)) \).

\[
\begin{align*}
\text{f(g(x))} &= f(x^2) = 2x^2 + 1 & \text{g(f(x))} &= g(2x + 1) = (2x + 1)^2 \\
\text{f(f(x))} &= f(2x + 1) = 2(2x + 1) + 1 & \text{g(g(x))} &= g(x^2) = (x^2)^2 \\
&= 4x + 3 & \text{= x}^4
\end{align*}
\]

We note that in general \( f(g(x)) \neq g(f(x)) \). Also note that in these composite functions the range of the ‘inside’ function becomes the domain of the outside function.

**Example 4**  Let \( f(x) = 2x \) and \( g(x) = x^2 \). If the domain is the set \( \{-1, 0, 1\} \) write down the elements of \( f(g(x)) \).

Substituting the domain values in function \( g \) produces:

\[
\begin{align*}
\text{g}(-1) &= 1, \text{g}(0) = 0 \text{ and } \text{g}(1) = 1.
\end{align*}
\]

These range values now become the domain values of function \( f \).

Substituting these domain values in function \( f \) produces:

\[
\begin{align*}
\text{f}(1) &= 2 \text{ and } f(0) = 0
\end{align*}
\]

Then \( f(g(x)) \) has elements \( (-1, 2), (0, 0) \) and \( (1, 2) \).

**Example 5**  If \( f(x) = x - 1 \) and \( g(x) = x^2 \) determine the value(s) of \( x \) for which \( f(g(x)) = g(f(x)) \).

\[
\begin{align*}
\text{f(g(x))} &= f(x^2) = x^2 - 1 \text{ and } g(f(x)) = g(x - 1) = (x - 1)^2 = x^2 - 2x + 1 \\
\text{Solving } x^2 - 1 &= x^2 - 2x + 1 \\
\text{ie } -1 &= -2x \\
\text{Hence: } x &= 1
\end{align*}
\]

Self-composition is, of course, also possible.

**Example 6**  Let \( h(x) = x^2 \). Obtain \( h(h(h(x)))) \).

\[
\begin{align*}
\text{h(h(h(x))))} &= h(h(x^2)) = h(x^4) = x^8
\end{align*}
\]

There are many applications of functions and function composition, especially in graphing. For example, to shift a graph with function equation \( f(x) = x^2 + 1 \) two units to the left, we apply the function \( f(x + 2) \). To graph the function \( \sin 2x \), we first apply the function \( f(x) = 2x \) and use the values so obtained in the sine function \( g(x) = \sin x \).
Activity 2.3
1. A relation \( f \) has ordered pairs of the form \((x, f(x))\) where \( f(x) = 3x - 1 \).
   Find the values of \( a, b \) and \( c \) if \( f = \{(1, a), (2, b), (c, -1)\} \).

2. If \( f(x) = x + 5 \), find the following:
   a) \( f(1) \)  
   b) \( f(3) \)  
   c) \( f(0) \)  
   d) \( f(-3) \)

3. If \( g(x) = x^2 + 2x + 1 \), find the following:
   a) \( g(1) \)  
   b) \( g(3) \)  
   c) \( g(0) \)  
   d) \( g(-3) \)

4. If \( h(x) = x^2 + 2 \) and \( g(x) = 2x \), find the following:
   a) \( h(g(1)) \)  
   b) \( g(h(3)) \)  
   c) \( g(g(0)) \)  
   d) \( g(g(g(g(1)))) \)

5. If \( r(x) = 2x^3 - 3x + 6 \) and \( s(x) = 4x + 6 \), calculate \( 2r(-4) + 7s(-2) + 3 \).

6. If \( f(x) = x^2 \) and \( g(x) = x + 2 \), determine the values of \( x \) for which \( f(g(x)) = g(f(x)) \).

2.4 Types of functions
We have already seen that by using the Cartesian Plane it is possible to represent a functional relationship by means of a graph. When graphing a function the notation \( y = f(x) \) is frequently used.

The graph of a function defined by \( y = f(x) \) consists of all ordered pairs \((x, y)\), or points, that satisfy the given equation. The points may form a line or a curve.

There are literally hundreds of functions that have special names and properties and if you continue to study Mathematics you will encounter many different types. However, in the remainder of this Section, we shall concentrate on three special types of functions and their graphs. These functions are listed in the following table.

<table>
<thead>
<tr>
<th>Name of function</th>
<th>Equation</th>
<th>Name of graph</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>( y = mx + b )</td>
<td>Line</td>
<td>( x ) to the power 1</td>
</tr>
<tr>
<td>Quadratic</td>
<td>( y = ax^2 + bx + c )</td>
<td>Parabola</td>
<td>( x ) is squared</td>
</tr>
<tr>
<td>Reciprocal</td>
<td>( y = \frac{k}{x} )</td>
<td>Hyperbola</td>
<td>( x ) is in the denominator</td>
</tr>
</tbody>
</table>
In the table the equations are written in a **standard form** but sometimes that is not the case and you have to carry out some simple algebra to rewrite the equations in standard form so that they can be identified. Let’s study a couple of examples.

**Example 1**  What type of graph is represented by the equation 2x + y = 8?

By transposing the term 2x, this equation can be re-written as y = −2x + 8 which means that a straight line would result on graphing.

**Example 2**  What type of equation is represented by xy = −9?

Cross multiplying x produces  \( y = \frac{-9}{x} \), which is a reciprocal function.

**Example 3**  Is the equation \( x^2 + y^2 + 2x = 9 \) the equation of a parabola?

At a first glance this appears to be the case but we note that the y is also squared. Thus the equation is not that of a parabola. In fact, the given equation would graph into a circle.

**Activity 2.4**

Identify each of the following as representing either a line (L), a parabola (P), a hyperbola (H) or neither (N).

a)  \( 2x + 4 = y \)  

b)  \( 2xy = 8 \)  

c)  \( x^2 - y^2 = 8 \)  

d)  \( 2x^2 - 7x - 6 = y \)  

e)  \( y = \frac{2x}{3} \)  

f)  \( x = -y + 7 \)  

**2.5 Drawing a linear function**

The linear function graphs into a line. Because a line is completely determined by two points we need only to find the coordinates of two points and connect them up. To be sure that we have not made a mistake, it is worthwhile to use a third point as a check.

When we select the points we generally select ‘easy points’. Those are points where one of x or y is 0; they are the **intercept** (with the axes) points.

You will find it convenient to set up a coordinate table as shown.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 1  Graph the line: $y = x + 1$.

Example 2  Graph the line: $y = -2x - 1$.

In your algebra course you have no doubt learnt how to solve simultaneous linear equations algebraically. A graphical approach can also be used. The intersection point of the lines (the graphs of the two linear equations), will then correspond to the solution.

An example will illustrate how simultaneous linear equations can be solved graphically.

Example 3  Solve $y = 2x - 4$ and $y = -x + 5$ simultaneously by graphical means.

$$
\begin{array}{c|c}
 x & y \\
 0 & -4 \\
 2 & 0 \\
 1 & -2 \\
\end{array}
\quad
\begin{array}{c|c}
 x & y \\
 0 & 5 \\
 5 & 0 \\
 2 & 3 \\
\end{array}
$$
The solution is (3, 2).

Activity 2.5
1. Using three points and a table, graph the lines with the following equations.
   a) \( y = x - 1 \)  
   b) \( y = 2x - 1 \)  
   c) \( x + y = 3 \)  
   d) \( x - 5y = 10 \)

2. Graphically solve the following pairs of simultaneous equations:
   a) \( y = x - 1 \) and \( y = 2x + 3 \)  
   b) \( y = 2x + 3 \) and \( y = x + 1 \)  
   c) \( y = -x + 5 \) and \( y = 2x - 1 \)  
   d) \( x + y = 7 \) and \( x - y = 1 \)

2.6 Using the gradient

You will have noticed that the lines in the examples of the previous topic have a different slope. In Example 1 the line slopes upwards or rises. It has what is called a positive gradient. Positive because when \( x \) values increase, so do \( y \) values. The situation is reversed in Example 2. Here the line slopes downwards or falls. Thus when \( x \) values increase, the \( y \) values decrease; the line has a negative gradient.

The symbol for gradient is \( m \). Because gradients are used in calculating the equations of trend lines, an important practical topic which will be dealt with later, it is important that you can calculate \( m \) values. Before we show how this is done let us give you the formal mathematical definition of gradient.
The gradient \((m)\) of a line is defined as:

\[
m = \frac{\text{rise}}{\text{run}}
\]

or \(m = \frac{\text{difference in } y \text{ values}}{\text{difference in } x \text{ values}}\)

or \(m = \frac{y_2 - y_1}{x_2 - x_1}\)

Finding the gradient is a matter of selecting two points on the graph and using them in one of the formulae above.

**Example 1** Calculate the gradient of the line shown.

Taking the points \((0, -1)\) and \((1, 1)\), we clearly see that the line has a rise of 2 for a run of 1. Hence the gradient is 2.

**Example 2** What is the gradient of this line?

Taking the points \((0, 2)\) and \((4, 0)\), the line rises (−2) units (ie fall 2 units) for a run of 4 units. Hence \(m = \frac{-2}{4} = -0.5\).
Example 3  Calculate the gradient of the line passing through (−5, 9) and (2, −7).

Here it is best to use the formula with \( x_1 = -5 \), \( y_1 = 9 \) and \( x_2 = 2 \) and \( y_2 = -7 \).

Then \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 9}{2 - (-5)} = \frac{-16}{7} = -2.29 \) rounded to two dp.

Note that there are two special cases:

• a horizontal line which has zero gradient
• a vertical line which has a gradient that is not defined.

Example 4  Graph the lines \( y = 3 \) and \( x = -2 \) on the same diagram.

In the example, the line \( y = 3 \) is a horizontal line. Because the line does not rise or fall its gradient is 0. Notice that in the table for \( y = 3 \), no matter what the value of \( x \) is, the \( y \) value is always 3. You may wish to check that by using the formula, you will obtain \( m = 0 \).

The line \( x = -2 \) is a vertical line with undefined gradient. The gradient is undefined because it is infinitely large. Imagine a line with gradient of, say, 50 000. This means that for every unit increase in the positive \( x \) direction, the \( y \) value increases by 50 000 units. This line must be nearly vertical but we can even make it ‘more vertical’. In fact, the gradient value increases without bound, the more parallel to the \( y \) axis the line becomes. When it is actually vertical, the gradient value cannot be given. Hence it is undefined.

Notice that in the table for \( x = -2 \), no matter what \( y \) value is chosen, the \( x \) value is always \( -2 \). Using the formula in this example results in 0 as the denominator. You may recall that dividing by zero is not possible, providing another reason why vertical lines have gradients that are not defined.

There are various ways to represent the equation of a line but the most useful way is by means of the gradient form.
You will recall that a line is determined by two points. In the following diagram, two special points are chosen. The first point is a point on the $y$ axis represented by $(0, b)$. The value $b$ is called the y intercept of the line. The second point is a general point which we call $(x, y)$.

![Diagram of a line with points](image)

Now we can determine the gradient of the line: 

$$m = \frac{y - b}{x - 0} = \frac{y - b}{x}$$

By cross multiplication we obtain $mx = y - b$, and by transposition we obtain the gradient form:

$$y = mx + b$$

Thus knowledge of the gradient and the y intercept enables us to write down the equation of the line immediately.

**Example 5** Write down the equation of the line through $(0, -5)$ and with gradient 4.

Here $m = 4$ and $b = (-5)$. Hence the equation of the line is $y = 4x - 5$.

**Example 6** What is the gradient and y intercept of $2x + y = 6$?

Here we must first transpose the $2x$ term. We obtain $y = -2x + 6$.

Thus the gradient of this line is $(-2)$ and the y intercept is 6.

Because a line is determined by two points, it is obvious that we should be able to use any two points to determine the equation of the line even if one is not the y intercept point.

The procedure is as follows:

- Obtain the value of $m$ first.
- Use this value and the coordinates of either of the two points and substitute into $y = mx + b$ to evaluate $b$.
- Write the equation as $y = mx + b$ using your values of $m$ and $b$. 
Example 7  Obtain the equation of the line through (2, 5) and (−4, 17).

Here \( m = \frac{17 - 5}{-4 - 2} = \frac{12}{-6} = -2 \)

Using (2, 5), we obtain \( 5 = -2(2) + b \) or \( 5 = -4 + b \)

Hence \( b = 9 \).

Thus the equation is \( y = -2x + 9 \).

Activity 2.6
1. Find the equation of the line with gradient 2, passing through the origin.
2. What is the equation of the line with gradient (−10) and y intercept 8?
3. Obtain the gradient and y intercept of the line with equation \( 2x - y = 7 \).
4. What are the gradients and y intercepts of lines A and B for the following?
   Write down their equations.
5. Determine the gradient of the line through (−1, 5) and (5, 8).
6. Draw the graph of the line that has a gradient of −3 and passes through the point (1, 2).
7. Obtain the equation of the line through (2, −9) and (3, 8).
8. Two lines that are parallel must have the same gradient. Using this fact, obtain the equation of the line parallel to \( y = -x + 7 \) and passing through the point (1, 3).
9. If two lines with respective gradients \( m_1 \) and \( m_2 \) are perpendicular then \( m_1m_2 = -1 \).
   Obtain the equation of the line that is perpendicular to the line \( y = -2x + 1 \) and passing through the point (2, −1).
10. Obtain the equation of the line parallel and perpendicular to \( 2x - y = 3 \) and passing through the point (2, 2).
2.7 Practical linear function problems

In this topic we take a look at practical ‘word’ problems in which the variables are connected by a linear function of the form $y = mx + b$.

The methodology is best illustrated by means of examples.

Example 1 A car salesman is paid a basic wage of $1400 a week and a commission of $200 per car sold.

Express the weekly wage of the salesman ($y$) in terms of the number of cars ($x$) sold and draw a graph of $y$ against $x$. If the salesman had an income of $2000 in one particular week, how many cars did he sell in that week?

If he does not sell any cars in a week, that is if $x = 0$, he still earns $1400. Thus the $y$ intercept is:

$$b = 1400.$$  

Further, for every extra car he sells, he received $200. Thus for an increase in $x$ of one unit, $y$ increases 200. Hence the gradient is:

$$m = 200.$$  

The appropriate equation then becomes $y = 200x + 1400$ and the graph is drawn below. Notice that the graph starts at $x = 0$; you obviously cannot sell a negative number of cars!

We can calculate how many cars he sells in the week he earns $2000 by solving $2000 = 200x + 1400$. Alternatively we can use the graph and move sideways from $y = 2000$ until we ‘hit’ the graph and find that the corresponding $x$ value is 3. Thus he sold 3 cars that week.

In many problems, it is more useful to use letters other than $x$ and $y$. A graph does not always have to be drawn.
Example 2

An apple farmer has a certain number of trees. Unfortunately, a disease starts killing the trees and they die off at a constant rate. After 3 weeks he has 925 left, while after 10 weeks only 750 are still alive.

By expressing the number of trees alive \((n)\) in terms of the number of weeks \((w)\), determine how many trees the farmer had originally.

There are two ordered pairs given \((3, 925)\) and \((10, 750)\). The gradient of the line through these points is:

\[
m = \frac{750 - 925}{10 - 3} = \frac{-175}{7} = -25
\]

This means that 25 trees die off per week.

Let there be originally \(b\) trees. Then, using \(m = -25\) we obtain:

\[n = -25w + b.\]

Substituting a point, say \((10, 750)\), produces: \[750 = -25(10) + b\]

or \[750 = -250 + b.\]

Solving gives \(b = 1000\).

Thus the farmer had 1000 trees originally.

In many practical problems we don't have exact data points but points that have been estimated. This is typically the case in experiments where we have to rely on the accuracy of our measuring instruments to obtain information. This type of data is called **empirical** data.

For example, a biologist may have experimental data about the amount of fertiliser used on peas and pea production. She may plot the data points and suspect that the two variables involved are linearly related because the plotted points lie nearly on a straight line. Of interest to her would be to find the relationship equation and use this line, called the **line of best fit** in this context, to make predictions about how much fertiliser should be used to produce a certain crop of peas.

The method we shall use here is:

- Use two points that are not 'outliers.'
- Proceed as before to find the gradient and y intercept and hence the equation of the line of best fit.

Note that there are more accurate methods available. In fact there are methods that use all the points, not just two. For the moment, however, we shall use the two-point method.
Example 3  The biologist collected the following empirical data.

<table>
<thead>
<tr>
<th>Amount of fertiliser (x)</th>
<th>No. of peas produced (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 g</td>
<td>200</td>
</tr>
<tr>
<td>10 g</td>
<td>325</td>
</tr>
<tr>
<td>20 g</td>
<td>490</td>
</tr>
<tr>
<td>40 g</td>
<td>510</td>
</tr>
</tbody>
</table>

Find the equation of the line of best fit and predict the number of peas produced when 15 g of fertiliser is applied.

Plotting the data clearly shows that the last point is an outlier. That point should not be used.

Using \((4, 200)\) and \((20, 490)\), we obtain \(y = 19.3x + 103.3\).

Plotting the line of best fit we obtain the following diagram.
Looking at a highly magnified section of this graph shows that 15 g of fertiliser is likely to produce 393 peas. This can be confirmed by calculation since when $x = 15$, $y = 19.3(15) + 103.3 = 392.8$.

In the last example it is clear that the amount of peas produced ‘tapers off’ when more fertiliser is applied. Indeed, we have to be careful that we do not extrapolate outside the given interval of data.

The line of best fit equation we have derived is really only valid for amounts of fertiliser in the interval from 0 g to 20 g and even near the 20 g level, there may be some doubt.

Now, imagine that we collect experimental data on two variables which when plotted do not produce points that lie approximately on a straight line. Then we cannot use the procedure outlined above unless we are able to transform the data to linear form first. This method involves the following steps:

- Guess the type of equation.
- Rewrite the ‘guessed’ equation in linear form by using an appropriate substitution.
- Use this substitution to transform the original data.
- Plot the data and check that a straight line results (apart from small irregularities).
- Determine the gradient and $y$ intercept.
- Use these values to write the ‘guessed’ equation.

Generally the most difficult part of this method is to find the appropriate substitution. Before doing a complete problem, we show an example on how to obtain an appropriate substitution.

**Example 4** Use an appropriate substitution to transform the equation $y = ax^2 + bx$ into linear form.

Let $y = ax^2 + bx$ and divide by $x$: $\frac{y}{x} = ax + b$.

Let $Y = \frac{y}{x}$, then the linear equation becomes $Y = ax + b$. 
If, in this last example, ordered pairs of the form \((x, y)\) were available, the \(y\) values would need to be divided by the corresponding \(x\) values to produce \(Y\) values and hence ordered pairs of the form \((x, Y)\).

Then we would check that the points would approximate a straight line from which the gradient and \(y\) intercept could then be determined. Obviously, the gradient would correspond to \(a\) and the \(y\) intercept to \(b\). Then the equation would be determined.

**Example 5**  
In an experiment the following readings were obtained:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0.1</th>
<th>0.5</th>
<th>1.1</th>
<th>2.9</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>3.2</td>
<td>8</td>
<td>5</td>
<td>3</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Show that the relationship is of the form \(y = \frac{c}{x} + k\) and find \(k\) and \(c\).

If \(y = \frac{c}{x} + k\), then on multiplying both sides by \(x\), we obtain:

\[yx = c + kx\]  

or

\[Y = kx + c\]

Thus all \(y\) values must be transformed to \(yx\). This produces the following table:

<table>
<thead>
<tr>
<th>(x)</th>
<th>0.1</th>
<th>0.5</th>
<th>1.1</th>
<th>2.9</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>3.2</td>
<td>4</td>
<td>5</td>
<td>8.7</td>
<td>15</td>
</tr>
</tbody>
</table>

After plotting we note that the points lie approximately on a straight line which verifies the relationship.

Because the line \(Y = kx + c\) has gradient 2 and \(y\) intercept 3, \(k = 2\) and \(c = 3\).
Activity 2.7

1. The cost of hiring a tool is made up of two parts. There is a fixed charge and a charge of $5 per hour of use. If it cost a total of $125 for 10 hours of use, determine the equation connecting cost \( c \) and the number of hours \( t \) the tool is used.

2. In a certain city, the taxi fare is made up of two parts, a fixed charge and a rate per kilometre. If a 10 km trip costs $32 and a 31 km trip costs $57.20:
   a) Express total cost \( c \) in terms of km \( k \) using \( m \) for gradient and \( b \) for vertical axis intercept.
   b) Draw a graph of total cost \( c \) against number of km \( k \).
   c) Use the graph to approximate how far, to the nearest km, you could travel for $40.
   d) Check your estimate by calculation.

3. The resistance of a component was measured at different temperatures in degrees Celsius. The following readings were taken.

<table>
<thead>
<tr>
<th>Temperature ( (T) )</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>130</th>
<th>150</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistance ( (R) )</td>
<td>55.4</td>
<td>58.6</td>
<td>61.8</td>
<td>64.8</td>
<td>66.0</td>
<td>60.5</td>
</tr>
</tbody>
</table>

   a) Graph \( R \) on the vertical axis against \( T \) on the horizontal axis.
   b) Draw a line through the points. It will not fit all points exactly.
   c) The line may be represented by the equation \( R = mT + b \). Use your graph to determine the values of \( m \) and \( b \).
   d) Use your graph to estimate the resistance when \( T = 105 \) °C.
   e) Check your estimate by calculation.
   f) Would it be feasible to estimate the resistance when the temperature is 200 °C?

4. The following values of \( p \) and \( v \) are believed to be connected by a law of the form \( p = \frac{a}{v} + b \). By plotting values of \( p \) against values of \( \frac{1}{v} \), verify this law and obtain suitable values of \( a \) and \( b \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>200</th>
<th>250</th>
<th>320</th>
<th>400</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v )</td>
<td>4</td>
<td>3.2</td>
<td>2.5</td>
<td>2</td>
<td>1.6</td>
</tr>
</tbody>
</table>

5. The power \( P \) generated by a wind-driven 30 cm two-bladed propeller depends on the wind speed \( v \). Data are shown in the following table:

<table>
<thead>
<tr>
<th>( v )</th>
<th>13</th>
<th>16</th>
<th>19</th>
<th>26</th>
<th>32</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>0.09</td>
<td>0.17</td>
<td>0.29</td>
<td>0.70</td>
<td>1.4</td>
<td>4.6</td>
</tr>
</tbody>
</table>

   a) By plotting \( P \) as a function of \( v^2 \) and of \( v^3 \), determine whether \( P \) is related \( v^2 \) or \( v^3 \).
   b) If the equation is of the form \( P = av^x \) \((x = 2 \text{ or } 3)\), determine the value of \( a \).
2.8 Drawing the quadratic function

The **parabola** is the result of drawing a curve defined by the quadratic function

\[ y = ax^2 + bx + c \]

The parabola is one of the more interesting and useful shapes that exist in our everyday life. When a ball is thrown it follows a parabolic path. Torches and telescopes use parabolic mirrors which ensures that light is reflected as a parallel beam. In order to ‘transport’ sound over long distances in a straight line, telephone signals are sent backward and forward using a parabolic dish. Even the Sydney Harbour Bridge is in the shape of a parabola as are all suspension bridges.

The coefficients \(a\), \(b\) and \(c\) determine the shape and positioning of the parabola.

The coefficient \(a\) determines the steepness of the curve. Note that \(a\) cannot be 0, otherwise a line will result. In fact if:

- \(a < 0\), the parabola slopes down from a maximum point
- \(a > 0\), the parabola slopes up from a minimum point.

The coefficient \(b\) can be shown to largely determine horizontal positioning while the coefficient \(c\) determines vertical positioning.

Some sketches will clarify this information.

---

A: \(y = x^2\) (\(a = 1\)) against B: \(y = -x^2\) (\(a = -1\))

**Note:** The parabola has inverted.
A: \( y = x^2 \) \( (a = 1) \) against B: \( y = 2x^2 \) \( (a = 2) \)

**Note:** the parabola has become steeper.

A: \( y = x^2 \) \( (b = 0) \) against B: \( y = x^2 + x \) \( (b = 1) \)

**Note:** The parabola has shifted to the left (and downwards).
All parabolas have a **line of symmetry**. It is a vertical line of the form \( x = k \), where \( k \) is a constant.

It can be shown that knowing the coefficients \( a \) and \( b \) completely determines the equation of the line. In fact its equation is:

\[
x = -\frac{b}{2a}
\]

After we obtain this value, we can substitute into the equation of the parabola to find the minimum or maximum value. This point is generally called the **turning point**.

**Example 1**  
Find the minimum turning point of: \( y = x^2 + 3x + 2 \).

Here \( a = 1 \), \( b = 3 \) and \( c = 2 \).

The line of symmetry has equation: \( x = -\frac{b}{2a} = -\frac{3}{2 \times 1} = (-1.5) \).

Substituting we obtain: \( y = (-1.5)^2 + 3(-1.5) + 2 = (-0.25) \)

Hence the minimum turning point is \((-1.5, -0.25)\).

To draw a parabola, it is generally sufficient to obtain the coordinates of the turning point and three or so other points on one side of the line of symmetry and then make use of the symmetric properties of the parabola.

Of course the coordinates of one point are always known and those are the coordinates of the \( y \) intercept of the parabola, ie \((0, c)\).
Example 2  Draw the parabola \( y = x^2 + 3x + 2 \).

Example 3  Draw the parabola \( y = 3 - 2x - x^2 \).

In order to determine \( a \), \( b \) and \( c \) we have to rewrite the equation in the usual format ie, \( y = ax^2 + bx + c \).

Interchanging terms on the right produces \( y = -x^2 - 2x + 3 \).

Hence \( a = (-1) \), \( b = (-2) \) and \( c = 3 \).

This parabola will have a maximum point since \( a < 0 \).

The line of symmetry is given by \( x = \frac{-b}{2a} = 1 \)

\[
\text{If } x = (-1), \ y = 4. \ \text{Hence the maximum turning point is at } (-1, 4)
\]
Activity 2.8
Graph the following parabolas.
1.  \( y = x^2 + 1 \)
2.  \( y = 3 - 2x^2 \)
3.  \( y = 2x^2 + x - 2 \)
4.  \( y = 6 - x - x^2 \)
5.  \( y = (x - 2)^2 \)

2.9 Simultaneous equations involving parabolas
In an earlier Section we solved simultaneously two linear equations. Here we have a look at how a linear (graphing into a line) and a quadratic equation (graphing into a parabola) can be solved simultaneously.
If we graph the equations simultaneously, it should be clear that three cases can be distinguished.

![Graphs of line and parabola]

Line and parabola do not intersect.
Line and parabola intersect in one point (they touch).
Line and parabola intersect in two points.

To find the simultaneous solutions is equivalent to finding the intersection points. We can use algebra or draw an accurate graph.
The process involved is illustrated in the following example, in which we obtain two distinct solutions.

Example
Solve \( y = 2x^2 + x - 3 \) and \( y = 2x - 2 \) simultaneously using:

a) algebra  
b) graphs.

a) Algebraically we equate the two equations:
\[ 2x^2 + x - 3 = 2x - 2 \text{ or } 2x^2 - x - 1 = 0 \]
Factorising: \( (2x + 1)(x - 1) = 0 \)
And hence \( x = -0.5 \) or \( x = 1 \).
The \( y \) values can be obtained by substituting the obtained \( x \) values in one of the equations (the linear one would be easiest).
If \( x = -0.5 \), \( y = -3 \) and if \( x = 1 \), \( y = 0 \).
b) Drawing the graphs:

The intersections (and therefore the solutions) are: 
\((-0.5, -3)\) and \((1, 0)\).

**Activity 2.9**

1. Consider \(y = x^2 - 2x - 1\) and \(y = -2x - 2\).
   By drawing sketches on the same diagram, determine which is correct.
   a) The graphs touch in one point.  
   b) The graphs intersect in two points.  
   c) The graphs do not intersect.  
   d) The graphs intersect in one point.

2. Consider \(y = -2x^2 - 4x - 1\) and \(y = 2x + 2\).
   By drawing sketches on the same diagram, determine which is correct.
   a) The graphs touch in one point.  
   b) The graphs intersect in two points.  
   c) The graphs do not intersect.  
   d) The graphs intersect in one point.

3. By algebraic means, solve simultaneously \(y = x - 4\) and \(y = -x^2 - 6x + 4\).

4. Draw, on the same diagram, accurate graphs of \(y = x^2 - 2x - 1\) and \(y = -x + 1\).  
   Use your graphs to find the simultaneous solutions.
2.10 Drawing the reciprocal function

Another important elementary function is the reciprocal function which has general equation:

\[ y = \frac{k}{x} \]

where \( k \) is a constant. Problems involving inverse variation in which one variable varies inversely with the other (e.g., speed and time to travel somewhere) can be described by this type of function.

The graph of the inverse function is a curve called the **hyperbola**. A hyperbola has two branches that are in opposite quadrants. In fact if:

- \( k > 0 \), the hyperbola has branches in quadrants I and III
- \( k < 0 \), the hyperbola has branches in quadrants II and IV.

A distinctive feature of the hyperbola is that the branches will get closer and closer to the axes making up the quadrant without intersecting them. We say that the axes are **asymptotes** of the graph.

**Example 1** Draw the graph of \( y = \frac{2}{x} \).

It is of interest to note that both the parabola and hyperbola are so-called **conic sections**. They are called conic sections because they can be obtained from the intersection of a cone with a plane tilted at various angles. Other conic sections are the circle and ellipse.

As with the line and parabola it is possible to change the position of the graph vertically and horizontally. It is also of course possible to intersect a hyperbola with a line, parabola or another hyperbola.

These ideas are explored in the next example.
Example 2  Graphically find the intersections of the hyperbola \( y = \frac{-1}{x - 2} \) with the line \( y = -x + 2 \).

Verify your results algebraically.

The hyperbola has \( k < 0 \) and will have branches in quadrants II and IV.

Also the vertical asymptote will have been shifted from \( x = 0 \) to \( x = 2 \).

The graphs intersect in the points (1, 1) and (3, -1).

Algebraically, we need to solve simultaneously \( y = \frac{-1}{x - 2} \) and \( y = -x + 2 \)

Equating \( y \) values gives:

\[
\frac{-1}{x - 2} = -x + 2
\]

Then \(-1 = (x - 2)(-x + 2)\) or \((-1) = -x^2 + 2x + 2x - 4\)

or \(x^2 - 4x + 3 = 0\)

or \((x - 1)(x - 3) = 0\)

Solving gives \( x = 1 \) and \( x = 3 \) with corresponding \( y \) values

\( y = 1 \) and \( y = (-1) \)

Hence the intersections occur at (1, 1) and (3, -1).

Activity 2.10

1. Draw the graphs of:
   a) \( y = \frac{2}{x} \)  
   b) \( y = -\frac{2}{x} \)

2. Draw an accurate sketch of:
   a) \( y = 1 + \frac{3}{x} \)  
   b) \( y = 2 - \frac{3}{x} \)
3. Graph the hyperbola defined by $xy = 5$.

4. What are the equations of the asymptotes of $y = -3 + \frac{2}{1-x}$?

5. On the same set of axes graph $y = \frac{1}{x}$ and $y = x^2 - 2x - 1$.

Find, accurately to two decimal places, the solution(s) to $\frac{1}{x} = x^2 - 2x - 1$. 
Assessment 2

1. A relation is defined as a set of ordered pairs \((x, y)\) such that \(y = x - 2\). If the domain is \(\{0, 1, 2, 3, 4\}\), list the range.

2. State the domain and range of the following functions:
   a) \(y = x^2\)  
   b) \(y = \sqrt{x}\)  
   c) \(y = \frac{1}{x}\)  
   d) \(y = \sqrt{9 - x^2}\)

3. Which of the following relations are functions?
   a) \(\{(1, 2), (2, 3), (3, 4)\}\)  
   b) \(\{(2, 2)\}\)  
   c) \(\{(1, 1), (1, 2), (2, 3)\}\)  
   d) \(\{(x, y) | y = 2x\}\)

4. Which relations are functions?
   a) Ellipse  
   b) Non vertical line  
   c) Hyperbola  
   d) Circle

5. Which relations are functions?
   a) Spiral  
   b) Point  
   c) Vertical line  
   d) Parabola

6. If \(f(x) = 2x^2 + 1\), evaluate:
   a) \(f(3) - f(2)\)  
   b) \(f(-1)\)  
   c) \((f(8))^2\)  
   d) \(f(f(-2))\)

7. If \(f(x) = x^2 + 1\) and \(g(x) = 2x\) and \(h(x) = x\), evaluate:
   a) \(f(g(x))\)  
   b) \(g(f(x))\)  
   c) \(h(g(x))\)  
   d) \(h(f(g))\)  
   e) \(h(2) + g(2) + f(2)\)  
   f) \(-f(3) + h(-1) - g(2)\)  
   g) \(f(f(2)) - g(g(2)) + h(h(2))\)  
   h) \(x\) for which \(f(x) = g(x)\)

8. Graph the line \(y = 3 - 2x\).

9. Find the gradient of the line \(5x + 3y = 1\).

10. What is the equation of the line with gradient 5 that passes through the point \((0, -2)\)?

11. Write down the equation of the line passing through \((5, -3)\) and \((-2, 4)\).

12. By graphical means, find the simultaneous solution to:
    \(y = 3x + 2\) and \(y = -x + 6\).
13. A table lamp manufacturer has a fixed weekly outlay independent of the number of lamps made. He sells 50 lamps in a certain week and incurs a loss of $200. In the following week he sells 60 lamps and makes a profit of $200.

a) Express his weekly profit ($p$) as a linear equation involving the number of lamps sold ($n$).

b) How much does he sell each lamp for?

c) What is his weekly outlay?

d) How many lamps should he sell to make a profit of $1000 per week?

e) What is his break-even point? (The break-even point is the number of lamps he should sell to ensure that he does not make a loss.)

f) Illustrate the information on a well-labelled diagram.

14. The acceleration, $a$, varies with time as shown in the following table:

<table>
<thead>
<tr>
<th>$t$</th>
<th>29.9</th>
<th>20.0</th>
<th>16.0</th>
<th>13.7</th>
<th>12.3</th>
<th>11.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.38</td>
<td>0.85</td>
<td>1.32</td>
<td>1.81</td>
<td>2.25</td>
<td>2.86</td>
</tr>
</tbody>
</table>

If the relationship between $a$ and $t$ is of the form $a = \frac{c}{t^2}$, determine the value of $c$.

15. Graph the parabola $y = 2x^2 - x - 1$ over the interval (domain) $-2 \leq x \leq 2$.

16. Graphically solve $-3x^2 - x + 3 = 0$.

**Hint:** Draw the parabola $y = -3x^2 - x + 3$ and determine where the parabola crosses the $x$ axis, i.e., where $y = 0$.

17. Make a sketch of the hyperbola $y + 2 = \frac{1}{x - 3}$ clearly indicating both asymptotes.

18. Simultaneously solve $y = x^2 - 2x - 3$ and $y = x + 1$. 
Answers to activities

Activity 2.1

1. a) Ordered  
   b) Element  
   c) Domain  
   d) Range

2. Range of Q = \{-2, -1, 0, 1, 2\}.

3. a) (2, 0), (2, 1), (2, 2), (1, 0), (1, 1)  
   b) (2, 1), (2, 2), (1, 0), (1, 1), (1, 2), (0, 0), (0, 1), (0, 2)  
   c) (2, 1), (2, 0), (1, 0)  
   d) (2, 0), (2, 1), (2, 2), (1, 0), (1, 1), (1, 2), (0, 0), (0, 1), (0, 2)

4. a) Domain: \(\mathbb{R}\), Range: \(\mathbb{R}\).  
   b) Domain: \(x = 2\), Range: \(\mathbb{R}\)  
   c) Domain: \(\mathbb{R} \geq 0\), Range: \(\mathbb{R} \geq 0\)  
   d) Domain \(\mathbb{R} \neq 0\), Range \(\mathbb{R} \neq 0\)

5. Domain: \(-2 \leq x \leq 2\), Range: \(-2 \leq y \leq 2\)

Activity 2.2

1. a) Yes  
   b) No  
   c) Yes  
   d) Yes

2. a) Yes  
   b) Yes  
   c) No  
   d) Yes

3. a) T  
   b) F  
   c) F  
   d) T

4. a) Yes  
   b) Yes  
   c) No  
   d) Yes

Activity 2.3

1. \(a = 2\), \(b = 5\) and \(c = 0\)

2. a) 6  
   b) 8  
   c) 5  
   d) 2

3. a) 4  
   b) 16  
   c) 1  
   d) 4
4. a) 6  
   b) 22  
   c) 0  
   d) 32  
5. 89  
6. $x = -0.5$  

**Activity 2.4**  
a) L  
b) H  
c) N  
d) P  
e) L  
f) L  

**Activity 2.5**  
1. a)  
   ![Graph](image_a)  
   
   b)  
   ![Graph](image_b)  
   
   c)  
   ![Graph](image_c)  
   
   d)  
   ![Graph](image_d)
Activity 2.6

1. \( y = 2x \)

2. \( y = -10x + 8 \)

3. Gradient: 2, \( y \) intercept: -7

4. Line A: \( m = 2, \ b = -1 \). Hence \( y = 2x - 1 \)
   
   Line B: \( m = -1, \ b = 3 \). Hence \( y = -x + 3 \)

5. \( m = 0.5 \)

6. \( x - y = 1 \)

\[ x = -\frac{y}{1} \]
7. \( y = 17x - 43 \)
8. \( y = -x + 4 \)
9. \( y = \frac{1}{2}x - 2 \)
10. Parallel: \( y = 2x - 2 \), Perpendicular \( y = -\frac{1}{2}x + 3 \)

**Activity 2.7**

1. \( c = 5t + 75 \)
2. a) \( c = mk + b \)
   
   ![Graph](image)
   
   c) Approximately 16.7 km
   
   d) \( 40 = 1.2k + 20 \)
   
   \[
   1.2k = 20 \\
   k = \frac{20}{1.2} = 16.67
   \]

3. a) and b)

   ![Graph](image)
   
   c) Depending on the points used, \( m = 0.15 \) and \( b \approx 46 \).
   
   d) \( \approx 62 \)
   
   e) Using solution from part c) \( R \approx 0.15 (105) + 46 \approx 62 \).
   
   f) No; too much outside the interval 90 – 150.
4. \( a = 800, \, b = 0 \)
5. a) \( P = av^3 \)
   b) \( a \approx 0.00004 \) depending on what points are used.

**Activity 2.8**

1.

2.

3.

4.

5.
Activity 2.9

1. c) 

2. b) 

3. \( x = -8 \), and \( y = -12 \) or, \( x = 1 \) and \( y = -3 \)
Solutions are: (2, -1) and (-1, 2).

Activity 2.10
Use quadratic, exponential, logarithmic and trigonometric functions and matrices.

2. a)

b)

5. $x = 2.55$ (2 dp).
Section 3 – Indices

3.1 Index notation

You are familiar with terms like $x^2$, $a^3$ and $y^4$. In these expressions the letter is called the **base** while the small superscript number is called the **index**. Sometimes an index is called an **exponent** or **power** and the word **logarithm** is used.

Formally we write: $a^n = a \times a \times a \times \ldots \times a$

$n$ factors of $a$

Note that there are a number of conventions on how to pronounce expressions involving index notation:

- $a^n$ is called the ‘$n$th power of $a$’ or ‘$a$ to the power of $n$’.
- The second and third powers are usually referred to as ‘squared’ and ‘cubed’. Thus $p^2$ is called ‘$p$ squared’ and $a^3$ is called ‘$a$ cubed’.
- An expression such as $(3a)^4$ is read as ‘$3a$ all to the power of four’. It means $(3a) \times (3a) \times (3a) \times (3a)$. On the other hand, $3a^4$ is read as ‘3 times $a$ to the power of four’. It means $3 \times a \times a \times a \times a$.

Also note that the first power is usually omitted. Thus $b^1$ is written as $b$. Further, be aware that when multiplying in algebra, the $\times$ is generally not written when brackets are used. Sometimes a dot (.) is used for the multiplication sign.

**Example 1** Express in index notation.

a) $b \times b \times b \times b$

b) $(-1) \times (-1) \times (-1)$

c) $10 \times 10 \times 10 \times 10 \times 10$

d) $(xy)(xy)(xy)(xy)(xy)$

e) $(x + y)(x + y)$

f) $5 \cdot 5 \cdot 5 \cdot 5$

Example 1

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^4$</td>
<td>$(-1)^3$</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$(xy)^5$</td>
</tr>
<tr>
<td>$(x + y)^2$</td>
<td>$5^4$</td>
</tr>
</tbody>
</table>

**Example 2** Write in expanded form.

a) $x^2 y^3$

b) $(3x)^2$

c) $-2x^4$

d) $(2y^2)^2$

e) $x \times x \times y \times y \times y$

f) $(3x) \times (3x)$

g) $-2 \times x \times x \times x \times x$

h) $(2y^2) \times (2y^2)$
Example 3  Evaluate:

\[
\begin{align*}
\text{a) } & \quad \frac{2^4}{8} = \frac{16}{8} = 2 \\
\text{b) } & \quad 2^2 \times \left( \frac{1}{5} \right)^2 = 2 \times \frac{1}{25} = \frac{2}{25} \\
\text{c) } & \quad (-2)^3 = -8 \\
\text{d) } & \quad x^2 \text{ if } x = -3 \\
\text{e) } & \quad \frac{2^4}{8} = \frac{16}{8} = 2 \\
\text{f) } & \quad 2^2 \times \left( \frac{1}{5} \right)^2 = 2 \times \frac{1}{25} = \frac{2}{25} \\
\text{g) } & \quad (-2)^3 = (-2) \times (-2) = -8 \\
\text{h) } & \quad x^2 = (-3) \times (-3) = 9
\end{align*}
\]

As we shall see later, powers of 10 are used in scientific calculations where we frequently deal with very small and very large numbers. Powers of 2 are important in computing while powers of \((-1)\) have applications in electronics. You may wish to check for yourself that \((-1)\) to an even power is positive and is negative when raised to an odd power.

Because powers of 2, 3 and 5 are frequently used in examples about indices, knowing the powers of these numbers, as displayed in the following table, is very beneficial.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(2^n)</th>
<th>(3^n)</th>
<th>(5^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>9</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>27</td>
<td>125</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>81</td>
<td>625</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>243</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>64</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Activity 3.1

1. Express in index form:

\[
\begin{align*}
\text{a) } & \quad n \times n \times n \\
\text{b) } & \quad (pg) \times (pg) \\
\text{c) } & \quad 0.1 \times 0.1 \times 0.1 \times 0.1 \times 0.1 \\
\text{d) } & \quad (a + b)(a + b)(a + b) \\
\text{e) } & \quad x \times x \times x \\
\text{f) } & \quad z^2 \times z^2
\end{align*}
\]

2. Write in expanded form:

\[
\begin{align*}
\text{a) } & \quad 23 \\
\text{b) } & \quad 4d^4 \\
\text{c) } & \quad (12k)^3 \\
\text{d) } & \quad -4s^3 \\
\text{e) } & \quad 4^2 \times z^3 \\
\text{f) } & \quad (-4s)^3
\end{align*}
\]

3. Express:

\[
\begin{align*}
\text{a) } & \quad 27 \text{ as a power of } 3 \\
\text{b) } & \quad 4096 \text{ as a power of } 4 \\
\text{c) } & \quad 1000 \text{ as a power of } 10 \\
\text{d) } & \quad 100000000 \text{ as a power of } 10 \\
\text{e) } & \quad 1024 \text{ as a power of } 2 \\
\text{f) } & \quad 59049 \text{ as a power of } 3
\end{align*}
\]
4. Calculate the following:
   a) \((-1)^2\)
   b) \((-1)^3\)
   c) \((-1)^{500}\)
   d) \((-1)^{501}\)
   e) \((-1)^{2n}\), \(n\) is a counting number.
   f) \((-1)^{2n+1}\), \(n\) is a counting number.

5. Evaluate:
   a) \((-2)^7\)
   b) \(-(-2)^7\)
   c) \(\frac{2^2}{5}\)
   d) \(\left(\frac{2^2}{5^2}\right)^2\)
   e) \(7^2 \times 10^3\)
   f) \(2^3 \times 3^3 \times 4^4\)

3.2 Multiplication rule

The notation \(a^n\) for \(n\) factors of \(a\), enables us to derive some simple rules for working with numbers written in index form. In this topic we derive the multiplication rule.

Consider the following examples and see if you can develop the multiplication rule yourself.

Example 1  Simplify: \(a^2 \times a^3\).
\[
a^2 \times a^3 = (a \times a) \times (a \times a \times a)
= a \times a \times a \times a \times a
= a^5
\]

Example 2  Simplify: \(12^4 \times 12^2\).
\[
12^4 \times 12^2 = (12 \times 12 \times 12 \times 12) \times (12 \times 12)
= 12 \times 12 \times 12 \times 12 \times 12 \times 12
= 12^6.
\]

From studying these examples you have probably discovered that, in general, the following rule holds:

**Multiplication rule**

\[a^m \times a^n = a^{m+n}\]
Even if you did not discover this rule yourself, you should verify it in the previous examples. Then, before trying this rule yourself in the Activity, you should study the following examples:

**Example 3**  
Simplify: \(e^4 \times e^7\).

\[
e^4 \times e^7 = e^{4+7} = e^{11}
\]

**Example 4**  
Simplify: \(a \times a^6\).

\[
a \times a^6 = a^1 \times a^6 = a^{1+6} = a^7
\]

**Example 5**  
Find \(x\) if \(a^{12} \times x = a^{16}\).

Require \(x\) such that \(12 + x = 16\).

Hence \(x = 4\).

**Example 6**  
Simplify: \(p^2x \times p^{-3x} \times p^2\).

\[
p^2x \times p^{-3x} \times p^2 = p^{2-3x+2} = p^{-x}
\]

**Example 7**  
Simplify: \((x^3y)^2 \times (x^2y)^2\).

\[
(x^3y)^2 \times (x^2y)^2 = (x^3y) \times (x^3y) \times (x^2y) \times (x^2y) = x^3 \times x^3 \times x^2 \times x^2 \times y \times y \times y \times y = x^{3+3+2+2} \times y^{1+1+1+1} = x^{10}y^4
\]

**Activity 3.2**

1. Simplify:
   a) \(n^3 \times n^2 \times n\)  
   b) \((pg)^6 \times (pg)^6\)  
   c) \(0.1^2 \times 0.1^4 \times 0.1^2 \times 0.1\)  
   d) \((a + b)^6 \times (a + b)^4\)  
   e) \(x^3 \times x^4 \times x^5\)  
   f) \(z^6 \times z^6\)

2. Simplify:
   a) \(6p^2 \times 9p^2\)  
   b) \((2d)^4 \times (3d)^2\)  
   c) \(7c^3 \times (-7c^4)\)  
   d) \(-4s^3 \times -3s^4\)  
   e) \(a^2 \times z^3 \times a^3 \times z\)  
   f) \((xy)^3 \times (x^2y)^3\)
3. Find $x$ if:
   
a) $2^x \times 2^6 = 2^{10}$ 
   
   b) $h^9 \times h^x = h^9$

   c) $(2i)^4 \times (2i)^x = (2i)^7$
   
   d) $3^2 \times 3^x \times 3^4 \times 3^5 = 3^{12}$

   e) $3^x \times 3^2 = 3^2$
   
   f) $3 \times 3^x \times 3^3 = 3^7$

3.3 Division rule

When numbers are expressed in index notation with the same base and these numbers are divided, a similar rule to the product rule applies.

Consider these examples.

**Example 1** Simplify: $a^5 \div a^3$.

$$a^5 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a} = a^2 \quad \text{(provided } a \neq 0)$$

**Example 2** Simplify: $10^4 \div 10^2$.

$$10^4 \div 10^2 = \frac{10 \times 10 \times 10 \times 10}{10 \times 10} = 10 \times 10 = 10^2$$

You have probably come to the conclusion that the division rule is as follows:

**Division rule**

$$a^m \div a^n = a^{m-n}$$

(provided $a \neq 0$)

**Example 3** Simplify: $10^{10} \div 10^7$ leaving your answer in index notation.

$$10^{10} \div 10^7 = 10^{10-7} = 10^3$$

**Example 4** Simplify: $k^5 \times k^6 \div k^3 \times k^4$.

$$\frac{k^5 \times k^6}{k^3 \times k^4} = \frac{k^{5+6}}{k^{3+4}} = \frac{k^{11}}{k^7} = k^{11-7} = k^4$$
Example 5  Find $x$ if $3^{12 + x} \div 3^7 = 3$.

Require $x$ such that $(12 + x) - 7 = 1$

or $5 + x = 1$

or $x = 1 - 5$

Hence: $x = (-4)$

Activity 3.3

1. Simplify:
   
   a) $n^3 \div n^2$
   
   b) $(pz)^8 \div (pz)^6$
   
   c) $r^5 \div r^2$
   
   d) $(a + b)^6 \div (a + b)^3$
   
   e) $p^3 \div p^3$
   
   f) $v^{231} \div v^{211}$

2. Simplify:
   
   a) $6p^2 \div 3p$
   
   b) $3^{m + n} \div 3^m$
   
   c) $7c^6 \div (-7c^4)$
   
   d) $\frac{3^4 \times 3^6}{3^8}$
   
   e) $\frac{2^{4x - 1}}{2^{3 - x}}$
   
   f) $\frac{14x^3y^7 \times 4x^5y^6}{7x^9y^8 \times 4xy^2}$

3. Find $x$ if:
   
   a) $2^x \div 2^6 = 2^{10}$
   
   b) $h^8 \div h^x = h^3$
   
   c) $2^2 = \frac{2^5}{2^x}$
   
   d) $j = \frac{x}{j}$
   
   e) $3^x \times 3^x \div 3^4 = 3^2$
   
   f) $\frac{2^{3x + 1}}{2^{3+x}} = \frac{2^{x+1}}{2^{1+x}}$
3.4 Negative and zero index rules

Let us start this topic by investigating an example.

**Example 1** Simplify: \(3^2 ÷ 3^4\).

\[
3^2 ÷ 3^4 = 3^{2-4} = 3^{-2}
\]

Alternatively:

\[
3^2 ÷ 3^4 = \frac{3^2}{3^4} = \frac{\beta \times \beta}{\beta \times \beta \times 3 \times 3} = \frac{1}{3^2}
\]

It appears that \(3^{-2}\) means the same as \(\frac{1}{3^2}\)

In fact the following rule holds:

**Negative index rule**

If \(a \neq 0\) then \(a^{-n} = \frac{1}{a^n}\)

**Example 2** Write \(7^{-5}\) with a positive index.

\[
7^{-5} = \frac{1}{7^5}
\]

**Example 3** Write \(\frac{1}{x^{-3}}\) with a positive index.

\[
\frac{1}{x^{-3}} = \frac{1}{1} = x^3
\]
Example 4  Calculate: \( \frac{1}{2^{-1} + 5^{-1}} \), leaving your answer as an improper fraction.

\[
\frac{1}{2^{-1} + 5^{-1}} = \frac{1}{\frac{1}{2} + \frac{1}{5}} = \frac{1}{\frac{5 + 2}{10}} = \frac{10}{10} = 1
\]

Sometimes when applying the division rule it is possible to obtain an index that is zero. Consider this example.

Example 5  Simplify: \( 5^4 + 5^4 \).

\[
5^4 + 5^4 = 5^4 \cdot 5^4 = 5^0
\]

Alternatively:

\[
5^4 + 5^4 = \frac{5^4}{5^4} = \frac{5 \times 5 \times 5 \times 5}{5 \times 5 \times 5 \times 5} = 1
\]

It appears that \( 5^0 \) means the same as 1.

In general:

Zero index rule
If \( a \neq 0 \) then \( a^0 = 1 \)

Example 6  Simplify: \( (x^2y^3z^{-5})^0 \).

\( (x^2y^3z^{-5})^0 = 1 \) since any number to the power zero equals 1.
Example 7  Simplify: \( \frac{5a^8 \times b^{15} \times c^0}{15a^7 \times b^{20} \times c^2} \).

\[
\frac{5a^8 \times b^{15} \times c^0}{15a^7 \times b^{20} \times c^2} = \frac{5}{15} \left( \frac{a^8}{a^7} \right) \left( \frac{b^{15}}{b^{20}} \right) \left( \frac{c^0}{c^2} \right) \\
= \frac{1}{3} \times a^1 \times \left( \frac{1}{b^5} \right) \times \left( \frac{1}{c^2} \right) \\
= \frac{a}{3b^5c^2}
\]

We finish this topic by making an observation about the logic of the rules discussed above.

When the table of powers of 2, 3 and 5 is modified the following table results.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(2^n)</th>
<th>(3^n)</th>
<th>(5^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>27</td>
<td>125</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>9</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(-1)</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{5})</td>
</tr>
<tr>
<td>(-2)</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{9})</td>
<td>(\frac{1}{25})</td>
</tr>
<tr>
<td>(-3)</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{27})</td>
<td>(\frac{1}{125})</td>
</tr>
</tbody>
</table>

Notice that while the power decreases by one, the next number is obtained by dividing the previous number by the base number. This is the reason that 1s appear in the row with \(n = 0\). Also, fractions result when numbers are raised to negative powers.
Activity 3.4

1. Simplify and write with positive indices.
   a) \( n^3 \div n^4 \)
   b) \( x^3 \div x^4 \)
   c) \( \frac{r^5}{r^{12}} \)
   d) \( (ab)^{-2} \)
   e) \( \frac{p^3}{p^2} \)
   f) \( \frac{1}{2^2} \)

2. Evaluate:
   a) \( 6p^2 \div 3p^2 \)
   b) \( (3m \cdot n)^0 \)
   c) \( 2^{2^{-2}} \)
   d) \( \frac{2^4 \times 3^9}{4^2} \)
   e) \( \frac{2^2 \times 2^{-3}}{2^{-8} \times 2^7} \)
   f) \( \frac{10^{-3} \times 10^{-4} \div 10^6}{10^4 \times 10^{-2}} \)

3. Find \( x \) if:
   a) \( 2x \times 2^6 = 1 \)
   b) \( h^8 + h^x = h^{-1} \)
   c) \( 2^{-2} = \frac{2^5}{2^x} \)
   d) \( 3^x = \frac{3^{-1}}{9} \)
   e) \( 3^x \times 3^x + 3^3 = 3^4 \)
   f) \( 2^0 = \frac{2^{-x+1}}{2^{1+x}} \)

3.5 Further index rules

Further index rules can be easily derived from examining some examples.

Example 1  Simplify: \( (x^4)^3 \).
\[
(x^4)^3 = (x^4) \times (x^4) \times (x^4)
\]
\[
= x^{4 \cdot 4 \cdot 4}
\]
\[
= x^{12}
\]

Example 2  Simplify: \( (5^{-2})^{-3} \).
\[
(5^{-2})^{-3} = \frac{1}{(5^{-2})^3}
\]
\[
= \frac{1}{5^{-2} \times 5^{-2} \times 5^{-2}}
\]
\[
= \frac{1}{5^{-6}}
\]
\[
= 5^6
\]
You probably discovered the following rule:

**Power of a power rule**
If \( a \neq 0 \) then \((a^m)^n = a^{mn}\)

**Example 3** Simplify: \((a^{-2})^{-3} \times (a^{-4})^2 + (a^2)^2\).

\[
(a^{-2})^{-3} \times (a^{-4})^2 + (a^2)^2 = a^6 \times a^{-8} + a^{-4}
\]

\[
= a^6 \times (-8) - (-4)
\]

\[
= a^2
\]

**Example 4** Find: \( x \) if \( m^{-20} = (m^x)^{-4} \).

Here \((-20) = -4x\)

Hence \( x = 5 \)

The rule investigated was called ‘power of a power’ rule. We shall now discuss the ‘power of a product’ rule.

Again, study some examples first.

**Example 5** Expand: \((a \times b)^4\).

\[
(ab)^4 = ab \times ab \times ab \times ab
\]

\[
= a \times a \times a \times a \times b \times b \times b \times b
\]

\[
= a^4 \times b^4
\]

**Example 6** Expand: \((2x^{-2})^2\).

\[
(2x^{-2})^2 = \frac{1}{(2x^{-2})^2}
\]

\[
= \frac{1}{(2x^{-2}) \times (2x^{-2})}
\]

\[
= \frac{1}{(2 \times 2) \times (x^{-2} \times x^{-2})}
\]

\[
= \frac{1}{4 \times x^{-4}}
\]

\[
= \frac{x^4}{4}
\]

You are probably now aware what the power of a product rule is.

**Power of a product rule**
If \( a \neq 0 \) and \( b \neq 0 \) then \((a \times b)^m = a^m \times b^m\)
Example 7  Simplify: \((p^2q^3)^2\).

\[
(p^2q^3)^2 = (p^2)^2 \times (q^3)^2 = p^4 \times q^6
\]

(Power of a product rule)

Example 8  Solve for \(x\): \((a^2)^x \times (a^3)^{x + 1} = a^3(a^6)^5\).

If \((a^2)^x \times (a^3)^{x + 1} = a^3(a^6)^5\)

then \(a^2x \times a^{3(x + 1)} = a^3 + 30\)

or \(a^{2x + 3(x + 1)} = a^{33}\)

Now \(2x + 3x + 3 = 33\)

From which \(5x = 30\)

Hence: \(x = 6\)

The last rule we discuss in this topic is the ‘power of a quotient rule’. This rule may be regarded as a special case of the ‘power of a product rule’

**Power of a quotient rule**

If \(a \neq 0\) and \(b \neq 0\) then \(\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\)

Example 9  Show the validity of the power of a quotient rule.

Simplify: \(\left(\frac{a}{b}\right)^n\).

\[
\left(\frac{a}{b}\right)^n = (a \times b^{-1})^n = a^n \times b^{-n} = \frac{a^n}{b^n}
\]

Example 10  Simplify: \(\frac{2.5^3}{0.5^3}\).

\[
\frac{2.5^3}{0.5^3} = \left(\frac{2.5}{0.5}\right)^3 = 5^3 = 125
\]
Example 11  Simplify and express with positive indices: \( \left( \frac{x^2 y^3}{z^2} \right)^{-2} \)

\[
\left( \frac{x^2 y^3}{z^2} \right)^{-2} = \frac{(x^2 y^3)^{-2}}{(z^2)^{-2}} \quad \text{(Power of a quotient rule)}
\]

\[
= \frac{x^{-4} y^{-6}}{z^4} \quad \text{(Power of a power rule)}
\]

\[
= \frac{x^4}{z^4 y^6}
\]

Activity 3.5

1. Simplify and write with positive indices:
   
   a) \((n^2)^4\)  
   
   b) \((x^6 + x^4)^2\)

   c) \(\left( \frac{x^2 z^5}{x^4} \right)^2\)

   d) \((a^2 b^{-3})^{-2}\)

   e) \(((c^{-1})^2)^2\)

   f) \((5z^2 \times 2z^{-1})^2\)

2. Evaluate:

   a) \((10^3)^2\)

   b) \((2^2 \times 5^0)^5\)

   c) \((5^{15} + 5^{10})^4\)

   d) \(\frac{(2^2 \times 3^1)^3}{(2^4 \times 3^{-1})^2}\)

   e) \([(-1)^4 \times 2^8 + 4^3]^3\)

   f) \(\frac{(10^{-3} \times 10^{-4} \times 10^6)^2}{(10^4 \times 10^2)^2}\)

3. What would be the powers of:

   a) \(x\) and \(y\) if \(\frac{14x^3 y^7 \times 4x^5 y^8}{5x^5 y^5 \times 4xy^{2^2}}\) is simplified?

   b) \(p\) and \(q\) if \(\frac{8 (pq^2)^5}{(2p^2 q^3)^2}\) is simplified?

   c) \(a\) and \(b\) if \(\frac{(14a^{-2}b^7)^2 \times 12ab}{6a^{-3}b^4}\) is simplified?
4. Solve for $x$:
   a) $(6^2)^x = 6^{12}$
   b) $(3 \times 7)^x = 21^0$
   c) $\left(\left(2^3\right)^3\right)^2 = 2^{26}$
   d) $\left(\frac{5^3}{5^2}\right)^x = 5^4$
   e) $(5^2 \times 2^x)^2 = 10^4$
   f) $2^{x-2} = 2^{2x+1}$

3.6 Fractional indices

In this Section we develop how a square root, cube root, etc. of a number may be expressed by means of indices.

As usual, we begin our investigation with an illustration example.

Example 1 Find the value of $9^{\frac{1}{2}}$.

We write 9 as $3^2$ and obtain:

$9^{\frac{1}{2}} = (3^2)^{\frac{1}{2}}$

$= 3^{2 \times \frac{1}{2}}$ (Power of a power rule)

$= 3$

In this example it appears that $9^{\frac{1}{2}} = 3$. This is confirmed by the multiplication rule because $9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9$. We also know that $\sqrt{9} = 3$. Thus $9^{\frac{1}{2}}$ means the same as $\sqrt{9}$.

Using $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a$, we can show that $a^{\frac{1}{2}}$ means the same as $\sqrt{a}$. In fact, in similar fashion it may be shown that $a^{\frac{1}{3}}$ means the same as $\sqrt[3]{a}$, $a^{\frac{1}{4}}$ means the same as $\sqrt[4]{a}$ etc.

This leads to the following rule:

**Fractional index rule**

If $a \geq 0$ then $a^{\frac{1}{n}} = \sqrt[n]{a}$

Thus the fractional index rule enables us to write expressions involving roots (sometimes called **radicals** or **surds**) as expressions with indices. This is a very important process in higher mathematics such as calculus. It is also very handy when doing calculations on the calculator.
Example 2  Evaluate: $\sqrt[5]{32}$.

$$\sqrt[5]{32} = 32^{\frac{1}{5}}$$

$$= (2^{5})^{\frac{1}{5}}$$

$$= 2$$

It is important to realise that all index rules we have developed also apply to fractional indices. This enables us to derive a more general fractional index rule.

**General fractional index rule**

If $a \geq 0$ then $a^\frac{m}{n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

The above rule follows from the power of a power rule in reverse:

$$a^\frac{m}{n} = (a^m)^\frac{1}{n} = \sqrt[n]{a^m}$$

Also

$$a^\frac{m}{n} = (a^m)^\frac{1}{n} = (\sqrt[n]{a})^m$$

Let us study some more examples.

Example 3  Write $\sqrt[7]{a^{-5}}$ with a positive index.

$$\sqrt[7]{a^{-5}} = a^{-\frac{5}{7}}$$

$$= \frac{1}{a^{\frac{5}{7}}}$$

Example 4  Write $\left(b^{\frac{2}{3}}\right)^{-5}$ in surd form.

$$\left(b^{\frac{2}{3}}\right)^{-5} = b^{-\frac{10}{3}}$$

$$= \sqrt[3]{b^{-10}}$$

Example 5  Evaluate: $243^{\frac{3}{5}}$.

$$243^{\frac{3}{5}} = (3^3)^{\frac{3}{5}}$$

$$= 3^3$$

$$= 27$$
Activity 3.6

1. Simplify if necessary and write in surd form:
   a) $x^{\frac{1}{4}}$
   b) $x^{\frac{2}{3}}$
   c) $x^{\frac{2}{5}}$
   d) $\left(\frac{1}{x^2}\right)^{\frac{1}{3}}$
   e) $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$
   f) $\left(\frac{1}{x^2} \div x^{\frac{1}{3}}\right) \times x^{\frac{1}{2}}$

2. Write in index form:
   a) $\sqrt{x}$
   b) $\sqrt[3]{x}$
   c) $\sqrt[3]{\sqrt{x}}$
   d) $\sqrt[5]{x^3}$
   e) $(\sqrt[3]{x^7})^3$
   f) $\frac{\sqrt{x} \times \sqrt[3]{x}}{\sqrt[2]{x}}$

3. Evaluate:
   a) $64^{\frac{2}{3}}$
   b) $16^{\frac{1}{2}}$
   c) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$
   d) $625^{\frac{3}{4}}$
   e) $100^{1.5}$
   f) $1000^{\frac{2}{3}}$

4. Solve for $x$:
   a) $(6^2)^x = 6^{\frac{1}{2}}$
   b) $8^{\frac{1}{3}} = \sqrt[3]{8}$
   c) $\left(\left(2\right)^2\right)^{\frac{2}{3}} = 2^3$
   d) $\left(\frac{5^3}{5^2}\right)^{\frac{1}{x}} = 5^{-2}$
   e) $3^{\frac{x}{2}} = 9$
   f) $\sqrt{3} \times (\sqrt{3})^x = \frac{(\sqrt{3}) \times \sqrt[3]{3}}{\sqrt[3]{3}^2}$

5. Express in simplest terms with positive indices:
   $\frac{\frac{3}{2}(p^2q^3r)}{\sqrt[4]{p^2q^{-1}}}$
3.7 Indicial equations

In an elementary way, you have already studied indicial equations because you have attempted questions in which the index was the unknown (usually \(x\)).

Indeed, indicial equations are those in which the pronumeral appears as the index like in \(2^x = 8\). These types of equations are part of a broader group of equations called exponential equations.

In indicial equations you should be able to obtain a solution by means of equating the indices of numbers which have the same base. This solution will be a rational number, ie a number that can be expressed in the form \(\frac{a}{b}\), \(b \neq 0\).

It is obvious that \(2^x = 8\) has the solution \(x = 3\), a rational number. On the other hand, \(2^x = 9\) cannot be solved by equating base numbers and will not have a rational solution. It is therefore not an indicial equation.

Let us see how indicial equations are solved.

**Example 1** Solve: \(3^{2k-5} = 1\).

If \(3^{2k-5} = 1\) then \(3^{2k-5} = 3^0\) or \(2k-5 = 0\) or \(2k = 5\) Hence: \(k = 2.5\)

**Example 2** Solve: \((2^5)^x = 4^{-2}\).

If \((2^5)^x = 4^{-2}\) then \(2^{5x} = (2^2)^{-2}\) or \(2^{5x} = 2^{-4}\) or \(5x = -4\) Hence: \(x = -0.8\)

**Example 3** Solve for \(t\): \(25^{t-1} = \frac{3\sqrt{5}}{1}\) or \(6(t - 1) = 1\)

If \(25^{t-1} = \frac{3\sqrt{5}}{1}\) then \(5^{2(t-1)} = 5^{\frac{1}{3}}\) or \(6t - 6 = 1\) or \(6t = 7\) or \(t - 1 = \frac{1}{3}\) Hence \(t = \frac{7}{6}\)

We finish with some difficult examples in which you have to use your factorising skills.
Example 4  Solve: $4^x - 3 \times 2^{x+2} + 32 = 0$ by letting $y = 2^x$.

If $4^x - 3 \times 2^{x+2} + 32 = 0$

then $(2^2)^x - 3 \times 2^x \times 2^2 + 32 = 0$

or $(2^2)^x - 12 \times 2^x + 32 = 0$.

Letting $y = 2^x$ produces $y^2 - 12y + 32 = 0$

or $(y - 8)(y - 4) = 0$

Hence: $(y - 8) = 0$ or $(y - 4) = 0$

From which $y = 8$ or $y = 4$

But $y = 2^x$

Thus $2^x = 8$ or $2^x = 4$

Hence: $x = 3$ or $x = 2$.

Some problems involving addition and subtraction can be solved by taking out a common factor.

Example 5  Solve: $2^{x+4} - 2^x = 60$.

If $2^{x+4} - 2^x = 60$

then $2^x(2^4 - 1) = 60$

or $2^x(16 - 1) = 60$

or $15 \times 2^x = 60$

or $2^x = 4$

Hence: $x = 2$

Activity 3.7

1. Solve the following:

   a) $3^{x+1} = 3^5$
   b) $5^{x-2} = 5^{-7}$
   c) $4^{2x} = 4^3$
   d) $2^{x-3} = 2$
   e) $6^{1-x} = 6^{-3}$
   f) $2^{m+3} = 8$
   g) $7^{3-x} = 49$
   h) $5^x = 1$
   i) $4^{1-2x} = 16$
   j) $3^{2x-5} = 1$
   k) $9^x = 27$
   l) $8^x = 4$
   m) $4^{x+2} = 8$
   n) $81^{1-2x} = 27$
   o) $4^x = 4 \sqrt{2}$
   p) $2^x = \sqrt{2}$
   q) $25^{x+1} = \sqrt{5}$
   r) $\frac{1}{8^4} = \sqrt{8}$
   s) $27^x = \frac{1}{3}$
   t) $9^x = \frac{\sqrt{3}}{3}$
Use quadratic, exponential, logarithmic and trigonometric functions and matrices

2. Solve the following by factorisation:
   a) $2^x - 2^{-x} = 1.5$
   b) $9^x - 10 \times 3^x + 9 = 0$
   c) $2^{x+7} - 3 \times 2^x = 250$
   d) $\frac{8^x - 8^{x-1}}{2^x} = 7$
   e) $\frac{9^{x+1} - 9^x}{3^{x+1}} = 8$
   f) $\frac{4^{x^2} + 4^x}{2^x} = 34$
   g) $3^x - 3^x = 54$
   h) $5^{2x} - 21 \times 5^x = 100$
   i) $4^x - 9 \times 2^x + 8 = 0$
   j) $9^x - 12 \times 3^x + 27 = 0$

3.8 Numerical and graphical methods

Recall that any equation in which the pronumeral appears as an index is called an exponential equation. We discovered that some exponential equations could be solved by equating base numbers. We called these equations indicial equations. But how do we solve ‘non-indicial’ equations such as $2^x = 18$? In this topic we investigate two ways of obtaining approximate solutions to such equations.

1. Numerical method

This method relies on a calculator and works by substituting values for the pronumeral to derive an approximate solution by trial and error. Solutions to any degree of accuracy can be found using this refinement method but the method is time consuming.

Consider the following example:

Example 1 Using the refinement technique, solve $2^x = 18$, accurately to one decimal place.

Since $2^4 = 16$ and $2^5 = 32$, the value of $x$ must lie between 4 and 5 but closer to 4. Trying 4.1 and 4.2, we obtain:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$2^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>17.148…</td>
</tr>
<tr>
<td>4.2</td>
<td>18.379…</td>
</tr>
<tr>
<td>4.15</td>
<td>17.753…</td>
</tr>
</tbody>
</table>

It is obvious that the solution is $x = 4.2$ correct to one decimal place. This is verified when $2^{4.15} = 17.753…$ is calculated because the solution must now lie between 4.15 and 4.2.
Example 2  Solve $3x = 2^{1-x}$ correct to two decimal places using the refinement technique.

The solution will be between 0 and 1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$3^x$</th>
<th>$2^{1-x}$</th>
<th>$3^x - 2^{1-x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0.5</td>
<td>1.732...</td>
<td>1.414...</td>
<td>0.317...</td>
</tr>
<tr>
<td>0.4</td>
<td>1.551...</td>
<td>1.515...</td>
<td>0.036...</td>
</tr>
<tr>
<td>0.39</td>
<td>1.534</td>
<td>1.526...</td>
<td>0.008...</td>
</tr>
<tr>
<td>0.38</td>
<td>1.518...</td>
<td>1.536...</td>
<td>-0.018...</td>
</tr>
<tr>
<td>0.385</td>
<td>1.526...</td>
<td>1.531...</td>
<td>-0.005...</td>
</tr>
</tbody>
</table>

The last column of the table shows that a change of sign appears between $x = 0.39$ and $x = 0.385$. This means that the solution, correct to two decimal places, is 0.39.

It must be emphasised that there are more sophisticated methods available to solve this type of equation, including the use of logarithms and calculus methods.

2. Graphical method

Another method of obtaining an approximate solution to an exponential equation that cannot be solved by means of equating base numbers is by graphing.

Here is an example.

Example 3  Solve $2^x = 3.5$ accurately to one decimal place.

The first step is to draw the graph of $y = 2^x$. A table is useful for this purpose.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0.25</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

Now the graph is drawn.
To solve the equation, it is necessary to find the $x$ value corresponding to $y = 3.5$. From the graph this value is approximately 1.8.

Hence the solution to $2^x = 3.5$ is 1.8 correct to one decimal place.

**Activity 3.8**

1. By the refinement method solve accurately to one decimal place:
   a) $2^x = 7$
   b) $3^x = 19$
   c) $3^x = 5^{1-x}$
   d) $2^{2x} = x + 2$

2. Use graph paper and draw an accurate sketch of $y = 3^x$ for values of $x$ ranging from (-2) to 2 (ie over the domain [-2, 2]).
   Use this graph to determine the approximate value of:
   a) $3^{1.5}$
   b) $3^{0.3}$

3. Use the graph drawn in question 2 above to find accurately to one decimal place the solution to:
   a) $3^x = 5$
   b) $3^x = 0.9$

**3.9 Formulae involving indices**

Frequently, formulae are encountered that involve indices. For example, in business studies, the formula $S = P(1 + r)^t$ is used to work out the accumulated sum $S$.

In mensuration the formula $V = \frac{4}{3} \pi r^3$ is used to calculate the volume of a sphere.

Other formulae may involve a square or a cube root like in the formula $T = 2 \pi \sqrt{\frac{L}{g}}$ which is used to calculate the time it takes to complete one swing of a pendulum.

In this topic we will take a look at how these kinds of formulae should be transposed. In the examples to follow, we will concentrate on the methodology of transposition; you are not required to know the meaning of the formulae, what units are involved or how to apply the formulae.

The general method involves isolating the variable to be transposed in an expression on one side of the equation and then negating whatever power the expression has. As usual, this method is best illustrated by means of examples.

**Example 1** Make $r$ the subject in the area formula for a circle $A = \pi r^2$.

If $A = \pi r^2$ then $\frac{A}{\pi} = r^2$ or $r^2 = \frac{A}{\pi}$

On taking square roots from both sides we obtain $r = \sqrt{\frac{A}{\pi}}$
Note that in spite of ending up with an equation of the form $x^2 = a^2$ which has two possible solutions, i.e., $x = a$ or $x = -a$, a negative solution is not appropriate in this example. This is because a length is always positive. Sometimes, however, two solutions are possible as in the following example.

**Example 2**  Let the equation of an ellipse be \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Express \( x \) in terms of the other variables.

If \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

then \( \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2} \)

or \( x^2 = a^2 \left( 1 - \frac{y^2}{b^2} \right) \)

Hence: \( x = \pm \sqrt{a^2 \left( 1 - \frac{y^2}{b^2} \right)} \)

In this last example, because coordinates can be positive or negative, the ± needed to be inserted.

**Example 3**  Make \( r \) the subject in \( V = \frac{4}{3} \pi r^3 \).

If \( V = \frac{4}{3} \pi r^3 \) then \( \frac{3V}{4\pi} = r^3 \) or \( r^3 = \frac{3V}{4\pi} \) on interchanging terms.

Hence: \( r = \left( \frac{3V}{4\pi} \right)^{\frac{1}{3}} \) or \( r = \sqrt[3]{\frac{3V}{4\pi}} \)

With odd powers no decisions about the validity of two answers need to be made. The same applies to problems involving fractional indices or surds. Consider this example:

**Example 4**  Let \( T = 2\pi \sqrt{\frac{L}{g}} \). Transpose the formula to make \( L \) the subject.

If \( T = 2\pi \sqrt{\frac{L}{g}} \)

then \( \frac{T}{2\pi} = \sqrt{\frac{L}{g}} \) or \( \frac{L}{g} = \left( \frac{T}{2\pi} \right)^2 \)

Squaring both sides produces \( \frac{L}{g} = \left( \frac{T}{2\pi} \right)^2 \) from which \( L = g \left( \frac{T}{2\pi} \right)^2 \) is derived.
Sometimes two procedures need to be employed.

Example 5  Let \( S = 4\pi r \sqrt{\frac{R^2 + r^2}{2}} \). Make \( R \) the subject.

If \( S = 4\pi r \sqrt{\frac{R^2 + r^2}{2}} \) then \( \frac{S}{4\pi r} = \sqrt{\frac{R^2 + r^2}{2}} \)

Squaring both sides gives: \( \frac{S^2}{16\pi^2 r^2} = \frac{R^2 + r^2}{2} \) or \( \frac{S^2}{8\pi^2 r^2} = R^2 + r^2 \)

Interchanging sides and subtracting \( r^2 \) from both sides leaves:

\[ R^2 = \frac{S^2}{8\pi^2 r^2} - r^2 \]

Finally, taking the square root from both sides (negative answer not appropriate) gives us the final answer:

\[ R = \sqrt{\frac{S^2}{8\pi^2 r^2} - r^2} \]

Activity 3.9

1. Make \( I \) the subject of the formula \( P = I^2 R \).
2. Make \( r \) the subject in the formula \( V = \pi r^2 h \).
3. If \( v^2 = u^2 + 2as \), express \( u \) in terms of the other variables.
4. Let \( d = \sqrt{\frac{d + e}{z}} \). Make \( e \) the subject.
5. Transpose the formula \( V = \sqrt{\frac{\pi d^2 h}{16}} \) to make \( h \) the subject.
6. Suppose \( T^2 = kS^3 \) where \( k = 109.23 \).

Transpose the formula first to make \( S \) the subject and then find \( S \), correct to two decimal places if \( T = 11.86 \).
3.10 Scientific and engineering notation

In technology, very large and very small numbers are frequently used. The speed of light, for example, is 290 000 000 m/s. On the other hand, the mass of a proton is 0.000 000 000 000 000 000 001 65 g.

It is obviously convenient to express such numbers in a shorter way. One method is to write the number in scientific notation. Another, related method, is to use engineering notation.

1. Scientific notation

In scientific notation, a number is written as the product of a decimal number between 1 and 10 (called the mantissa) multiplied by an integer power of 10. This notation is also called standard notation.

Example 1 Which have been written in scientific notation?

a) $2.1 \times 10^3$  b) 1.94

c) $3.43 \times 10^{1.5}$  d) $21.7 \times 10^{-1}$

e) $6.781 \div 10^{-35}$  f) $2.987456 \times 10^{30}$

a) Yes.  b) Yes. Power of 10 is 0.

c) No. Power not an integer.  d) No. Decimal not between 1 and 10.

e) No. Not a product.  f) Yes.

Converting between ordinary decimal notation and scientific notation is essentially a matter of counting the number of decimal places the decimal point needs to be shifted either to the left (resulting in positive powers of 10) or right (resulting in negative powers of 10).

The following examples reinforce this idea.

Example 2 Express each of the following in scientific notation.

a) 4 600 000  b) 0.000 000 076

a) $4.6 \times 10^6$  
(The decimal point has shifted 6 places to the left.)

b) $0.000 000 76 = \frac{7.6}{1000000} = 7.6 \times 10^{-7}$  
(The decimal point has shifted 7 places to the right.)
Example 3  Express each of the following in ordinary decimal notation.

a) \(2.5 \times 10^{-3}\)

\[= \frac{2.5}{10^3} = \frac{2.5}{1000} = 0.0025\]

b) \(5.99 \times 10^4\)

\[= 5.99 \times 10 \times 10000 = 59900\]

2. Engineering notation

In engineering notation, a number is written as the product of a decimal number between 1 and 1000 multiplied by an integer power of 10 that is a multiple of 3.

Example 4  Which have been written in engineering notation?

a) \(22.1 \times 10^3\)
b) 194
c) \(13.43 \times 10^{1.5}\)
d) \(0.217 \times 10^9\)
e) \(16.781 \times 10^{-36}\)
f) \(221.456 \times 10^{19}\)

a) Yes.  b) Yes. Power of 10 is 0.
c) No. Power not an integer.  d) No. Decimal not between 1 and 1000.
e) Yes.  f) No. Power of 10 not a multiple of 3.

In converting numbers to engineering notation it is usually easiest to express the number in scientific notation first. If the power of 10 is not already a multiple of 3, the mantissa can be easily adjusted.

The following example will make this clear.

Example 5  Express each of the following in engineering notation.

a) \(46 600\)
b) \(0.000 000 43\)

a) \(46 600 = 4.66 \times 10 000 = 4.66 \times 10^4 = 46.6 \times 10^3\)

(The mantissa is multiplied by 10 to reduce the power by 1.)

b) \(0.000 000 43 = \frac{4.3}{10000000} = 4.3 \times 10^{-7} = 430 \times 10^{-9}\)

(The mantissa is multiplied by 100 to reduce the power by 2.)
Powers of 10 that are multiples of 3 have special prefixes and symbols which are listed in the following table.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Prefix</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{24}$</td>
<td>yotta-</td>
<td>Y</td>
</tr>
<tr>
<td>$10^{21}$</td>
<td>zetta-</td>
<td>Z</td>
</tr>
<tr>
<td>$10^{18}$</td>
<td>exa-</td>
<td>E</td>
</tr>
<tr>
<td>$10^{15}$</td>
<td>peta-</td>
<td>P</td>
</tr>
<tr>
<td>$10^{12}$</td>
<td>tera-</td>
<td>T</td>
</tr>
<tr>
<td>$10^9$</td>
<td>giga-</td>
<td>G</td>
</tr>
<tr>
<td>$10^6$</td>
<td>mega-</td>
<td>M</td>
</tr>
<tr>
<td>$10^3$</td>
<td>kilo-</td>
<td>k</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>milli-</td>
<td>m</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>micro-</td>
<td>μ</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>nano-</td>
<td>n</td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>pico-</td>
<td>p</td>
</tr>
<tr>
<td>$10^{-15}$</td>
<td>femto-</td>
<td>f</td>
</tr>
<tr>
<td>$10^{-18}$</td>
<td>atto-</td>
<td>a</td>
</tr>
<tr>
<td>$10^{-21}$</td>
<td>zepto-</td>
<td>z</td>
</tr>
<tr>
<td>$10^{-24}$</td>
<td>yocto-</td>
<td>y</td>
</tr>
</tbody>
</table>

The prefixes are often used in conjunction with engineering notation as illustrated in this example.

**Example 6** Write 25.5 nA in ordinary decimal notation.

$$25.5 \text{ nA} = 25.5 \times 10^{-9} \text{ A} = \frac{25.5}{1000\,000\,000} = 0.000\,000\,0255 \text{ A}$$

Note that you can enter numbers into a scientific calculator in either scientific or engineering notation. You can also use the calculator to convert between the different types of notation. You should check your calculator instruction booklet on how to do this.

We finish with an applied problem in which the calculator has been used.

**Example 7** Boyle’s Law states that, at a constant temperature, the volume of a mass of gas varies inversely with the pressure. Symbolically the law may be written as:

$$p_1 V_1 = p_2 V_2$$

If $V_1 = 2 \times 10^{-3}$, $p_1 = 2.513 \times 10^5$ and $p_2 = 4.013 \times 10^5$, calculate $V_2$.

State your answer in engineering notation correct to two decimal places.

If $p_1 V_1 = p_2 V_2$, then by cross multiplication $V_2 = \frac{p_1 V_1}{p_2}$.

Substituting: $V_2 = \frac{2.513 \times 10^5 \times 2 \times 10^{-3}}{4.013 \times 10^5} = 1.25 \times 10^{-3}$
Activity 3.10

1. Write the following using scientific notation:
   a) 980 000
   b) 0.003
   c) 8 000 000 000
   d) 0.000 000 000 000 012

2. Write the following using engineering notation:
   a) 0.000 009
   b) 879 000
   c) 0.004
   d) 1 789 000 000

3. Write in ordinary decimal notation:
   a) $977.5 \times 10^{-9}$
   b) $23.87 \times 10^{6}$
   c) 234 G
   d) 12.765 p

4. Use the calculator to evaluate the following expressions. Give your answer in scientific notation and to an accuracy of two decimal places.
   a) $56 534 \times 9 282$
   b) $998 390 \div 7 655$

5. Evaluate the following, writing your answer in engineering notation and accurate to two decimal places.
   a) $879 \times 987 \times 213$
   b) $\sqrt{765 876}$

6. Simplify $(3.98 \times 10^7) \times (2.54 \times 10^{-4})$ expressing your answer in:
   a) scientific notation.
   b) engineering notation.

7. Simplify $\frac{(13.48 \times 10^{-3}) + (12.34 \times 10^{-2})}{145 \times 10^{-9}}$ expressing your answer in:
   a) scientific notation.
   b) engineering notation.

8. The third law of planetary motion discovered by Johannes Keppler states that the ratio $k = \frac{r^3}{T^2}$ is the same for all planets where $r$ is the average radius of the orbit of the planet around the sun and $T$ is the time it takes to make one orbit.
a) Calculate accurately to two decimal places, the value of $k$ given that for Earth $r = 149 \times 10^6$ km and $T = 365.26$ days.

b) If for Venus $r = 108.2 \times 10^6$, calculate accurately to two decimal places the duration of its orbit around the sun.

c) For Pluto $T = 90474.9$ days. Calculate $r$ accurately to two decimal places and expressed in scientific notation.
Assessment 3

1. Express:
   a) \( nnnnnnnn \) as a power of \( n \)
   b) 4096 as a power of 4

2. Simplify:
   a) \( d^8 \times d^6 \)
   b) \( e^{2x} \times e^{-3x} \)
   c) \( \frac{3^9}{3^6} \)
   d) \( \frac{2^{4x-1}}{2^{3-x}} \)

3. Express without brackets:
   a) \((mn)^4\)
   b) \((2x)^3\)
   c) \(\left( \frac{3}{4} \right)^4\)
   d) \(\left( \frac{r}{st} \right)^n\)

4. Express:
   a) \((15)^4\) in terms of powers of 3 and 5
   b) \((24)^3\) in terms of powers of 2 and 3
   c) 0.6 in terms of powers of 3 and 5
   d) 1.8 in terms of powers of 3 and 5

5. Express as a single power of 3:
   a) \((3^2)^6\)
   b) \(81^{x+2}\)
   c) \(\sqrt{3}\)
   d) \(27^{\frac{1}{3}}\)

6. Express in simplest terms with positive indices:
   a) \(\frac{(x^3yz^2)^2}{(x^2y^2)^3}\)
   b) \(\frac{\sqrt[3]{(p^2q)^2r}}{\sqrt[3]{p^2q^{-1}}}\)

7. Find the value of:
   a) \(\frac{1}{2^{-1} + 4^{-1}}\)
   b) \(2^{0.5} \times 32^{0.5}\)

8. Evaluate:
   a) \((2x^3)^0\)
   b) \(\frac{(3^0 m^{-2} n^0)^2}{m^4}\)
9. Simplify:
   a) \( \frac{2x^2}{y^2} \times \left( \frac{1}{x^3 y^3} \right)^{-1} \)
   b) \( [(a + 1)^{-1} + 1] \div [(a + 1)^{-1} - 1] \)

10. By equating base numbers solve:
   a) \( 2^x = 64 \)
   b) \( 3^{x+2} = 81 \)
   c) \( 2^{-x} = 16 \)
   d) \( 5^{2x} = 125 \cdot 2x + 1 \)

11. Use the refinement method to obtain the solution to \( 2.5^x = 7 \) correct to two decimal places.

12. Draw the graph of the function \( y = \left( \frac{1}{2} \right)^x \), and use it to determine the solution to \( \left( \frac{1}{2} \right)^x = 3 \) correct to one decimal place.

13. Solve the equation \( 2^x - 2^{-x} = 1 \) by drawing \( y = 2^x \) and \( y = 2^{-x} + 1 \) on the same axes and observing where they intersect.
   **Hint:** Draw the graphs for \(-3 \leq x \leq 3\).

14. Transpose to make \( b \) the subject.
   a) \( I = \frac{1}{6} (a + b^3) \)
   b) \( d = \frac{12I}{\sqrt{b}} \)

15. Express in scientific notation:
   a) \( 345 \,000 \,000 \)
   b) \( 0.000 \,000 \,000 \,000 \,000 \,000 \,000 \,000 \,987 \)
   c) \( \sqrt{45636} \times 0.8765 \) to 2 dp
   d) \( (2.435 \times 10^{-9}) \div (1.98 \times 10^{-5}) \) to 2 dp

16. Express \( \sqrt{2.4 \times 10^{17}} \) in:
   a) engineering notation to 2 dp
   b) scientific notation to 2 dp
Answers to activities

Activity 3.1

1. a) \(n^3\) b) \((pq)^2\)
   c) \(0.1^5\) d) \((a + b)^3\)
   e) \(x^4\) f) \(z^4\)

2. a) \(2 \times 2 \times 2\) b) \(4 \times d \times d \times d \times d\)
   c) \(12k \times 12k \times 12k\) d) \(-4 \times s \times s \times s\)
   e) \(4 \times 4 \times z \times z \times z\) f) \((-4s) \times (-4s) \times (-4s)\)

3. a) \(3^3\) b) \(4^6\)
   c) \(10^3\) d) \(10^9\)
   e) \(2^{10}\) f) \(3^{10}\)

4. a) 1 b) \(-1\)
   c) 1 d) \(-1\)
   e) 1 f) \(-1\)

5. a) \(-128\) b) 128
   c) 0.8 d) 0.0256
   e) 49 000 f) 27 648

Activity 3.2

1. a) \(n^6\) b) \((pg)^{12}\)
   c) \(0.19\) d) \((a + b)^{10}\)
   e) \(x^{14}\) f) \(z^p + q\)

2. a) \(54p^4\) b) \(144d^6\)
   c) \(-49c^7\) d) \(12s^7\)
   e) \(a^2z^4\) f) \(x^5y^9\)
Section 3 Indices

3. a) \( x = 4 \)  b) \( x = 1 \)  
c) \( x = 3 \)  d) \( x = 1 \)  
e) \( x = 1 \)  f) \( x = 3 \)  

Activity 3.3

1. a) \( n \)  b) \((pz)^2\)  
c) \( r^3 \)  d) \((a + b)^3\)  
e) 1  f) \( y^{20} \)  

2. a) \( 2p \)  b) \( 3^n \)  
c) \( -c^2 \)  d) \( 3^2 \)  
e) \( 2^{5x - 4} \)  f) \( 2x^2y^6 \)  

3. a) \( x = 16 \)  b) \( x = 5 \)  
c) \( x = 3 \)  d) \( x = j^2 \)  
e) \( x = 3 \)  f) \( x = 1 \)  

Activity 3.4

1. a) \( \frac{1}{n} \)  b) \( x^{12} \)  
c) \( \frac{1}{r^7} \)  d) \( \frac{1}{(ab)^2} \)  
e) 1  f) \( 2^2 \)  

2. a) 2  b) 1  
c) 1.25  d) 1  
e) 1  f) \( \frac{1}{10^{15}} \)  

3. a) \( x = 6 \)  b) \( x = -9 \)  
c) \( x = 7 \)  d) \( x = -3 \)  
e) \( x = 2 \)  f) \( x = 0 \)
Activity 3.5

1. a) \( n^{12} \)  
   b) \( x^4 \)  
   c) \( x^{12}z^{10} \)  
   d) \( \frac{b^6}{a^3} \)  
   e) \( c^4 \)  
   f) \( 100z^2 \)

2. a) \( 1\,000\,000 \)  
   b) \( 1024 \)  
   c) \( 3125 \)  
   d) \( 60.75 \)  
   e) \( 4096 \)  
   f) \( 100 \)

3. a) 2 and 6  
   b) 1 and 4  
   c) 8 and \(-17\)  

4. a) \( x = 6 \)  
   b) \( x = 0 \)  
   c) \( x = 6 \)  
   d) \( x = 4 \)  
   e) \( x = 2 \)  
   f) \( x = -3 \)

Activity 3.6

1. a) \( \sqrt[4]{x} \)  
   b) \( \sqrt[3]{x^2} \)  
   c) \( \sqrt[5]{x^{-2}} \)  
   d) \( \sqrt[6]{x} \)  
   e) \( \sqrt[6]{x^5} \)  
   f) \( \sqrt[12]{x^5} \)

2. a) \( x^{\frac{1}{2}} \)  
   b) \( x^{\frac{1}{3}} \)  
   c) \( x^{\frac{1}{6}} \)  
   d) \( x^{\frac{7}{5}} \)  
   e) \( x^{\frac{7}{2}} \)  
   f) \( x^{\frac{7}{12}} \)

3. a) 16  
   b) 0.5  
   c) 0.75  
   d) 125  
   e) 1000  
   f) 0.01

4. a) \( x = 0.25 \)  
   b) \( x = 2 \)  
   c) \( x = 9 \)  
   d) \( x = -0.5 \)  
   e) \( x = 4 \)  
   f) \( x = \frac{5}{3} \)

5. \( \frac{11}{p^5}q^{\frac{11}{7}}r^3 \)
Section 3 Indices

Activity 3.7

1. a) \( x = 4 \)  
   b) \( x = -5 \)  
   c) \( x = 1.5 \)  
   d) \( t = 4 \)  
   e) \( x = 4 \)  
   f) \( m = 0 \)  
   g) \( x = 1 \)  
   h) \( x = 0 \)  
   i) \( x = -0.5 \)  
   j) \( k = 2.5 \)  
   k) \( x = 1.5 \)  
   l) \( x = \)  
   m) \( x = -0.5 \)  
   n) \( x = 0.125 \)  
   o) \( x = 1.25 \)  
   p) \( x = 0.5 \)  
   q) \( t = -0.75 \)  
   r) \( x = 2 \)  
   s) \( x = \frac{1}{3} \)  
   t) \( x = -0.25 \)  
   u) \( x = 4 \)  
   v) \( x = \frac{7}{6} \)  
   w) \( x = 1 \)  
   x) \( x = 2 \)  
   y) \( x = 1 \)  
   z) \( x = -0.25 \) 

2. a) \( x = 1 \)  
   b) \( x = 0 \) or \( x = 2 \)  
   c) \( x = 1 \)  
   d) \( x = 1.5 \)  
   e) \( x = 1 \)  
   f) \( x = 1 \)  
   g) \( x = 4 \)  
   h) \( x = 2 \)  
   i) \( x = 3 \) or 0  
   j) \( x = 1 \) or \( x = 2 \) 

Activity 3.8

1. a) \( x = 2.8 \)  
   b) \( x = 2.7 \)  
   c) \( x = 0.6 \)  
   d) \( x = 0.7 \) or \( -1.9 \)
2. 

\[ y = 2^x \]

\[ \begin{array}{c|c|c|c|c|c|c|c} 
\hline
x & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\hline
y & 0.5 & 1 & 2 & 4 & 8 & 16 & 32 \\
\hline
\end{array} \]

a) \( \approx 5.2 \)  

b) \( \approx 1.4 \)

3. a) \( x = 1.5 \)  

b) \( x = -0.1 \)

**Activity 3.9**

1. \[ I = \pm \sqrt{\frac{P}{R}} \]

2. \[ r = \frac{\sqrt{V}}{\pi h} \] (only positive values possible)

3. \[ u = \pm \sqrt{v^2 - 2as} \]

4. \[ e = zd^4 - d \]

5. \[ h = \frac{16V^2}{\pi d^4} \]

6. \( S = 1.09 \)
Activity 3.10

1. a) $9.8 \times 10^5$
   b) $3 \times 10^{-3}$
   c) $8 \times 10^{12}$
   d) $1.2 \times 10^{-17}$

2. a) $9 \times 10^{-6}$
   b) $879 \times 10^3$
   c) $4 \times 10^{-3}$
   d) $1.789 \times 10^{12}$

3. a) $0.000\ 000\ 977\ 5$
   b) $23\ 870\ 000$
   c) $234\ 000\ 000\ 000$
   d) $0.000\ 000\ 000\ 012\ 765$

4. a) $5.25 \times 10^8$
   b) $1.30 \times 10^2$

5. a) $184.79 \times 10^6$
   b) $875.14 \times 10^0 = 875.14$

6. a) $1.01 \times 10^{-1}$ (3 S.F.)
   b) $101 \times 10^{-3}$ (3 S.F.)

7. a) $9.44 \times 10^1$
   b) $94.4 \times 10^0 = 94.4$ (3 S.F.)

8. a) $2.48 \times 10^{19}$
   b) $226.03$ days
   c) $5.88 \times 10^9$ km
Section 4 – Logarithms

4.1 Definition of a logarithm

Consider the formula \( S = P(1 + r)^t \). This formula is used in financial mathematics to calculate the accumulated sum \( S \) when a principal amount \( (P) \) is invested at \( r \% \) interest per year over \( t \) years.

In this formula, \( S \) is the subject and a calculator can easily be used to calculate \( S \) on the basis of values of \( P \), \( r \) and \( t \).

When \( P \) needs to be calculated when \( S \), \( r \) and \( t \) are known, the formula must first be transposed to make \( P \) the subject. In fact \( P = \frac{S}{(1+r)^t} \).

When \( P \), \( S \) and \( t \) are known, we can also obtain by transposition an expression to calculate \( r \). This is a little more difficult and involves \( t \) as the root. In fact, \( r = \sqrt[t]{\frac{S}{P}} - 1 \).

But what should we do to get an expression for \( t \) when \( P \), \( S \), and \( r \) are known? In financial terms, this equates to finding out how long a principal amount \( P \) needs to be invested at \( r \% \) interest per year in order to grow to an accumulated sum of \( S \). This is a very important question in financial mathematics.

Thus we need an expression so that we can write \( t = \ldots \) Here we use logarithms.

Consider the exponential equation \( b^x = y \).

To change this expression and to make \( x \) the subject we write \( x = \log_b y \).

\( b^x = y \) is called the exponential statement or form and 
\( x = \log_b y \) is called the logarithmic statement or form.

In order to work with logarithms you should be able to move effortlessly between the two statements.

Example 1  \( 3^4 = 81 \) in logarithmic form.

Here the base is 3 (the small number under the log sign) and the index is 4.

Hence: \( 3^4 = 81 \) in logarithmic form is \( 4 = \log_3 81 \).

Example 2  \( \sqrt[4]{16} = 4 \) as a logarithmic statement.

Note that \( \sqrt[4]{16} \) can be written as \( 16^{\frac{1}{4}} \). Hence the exponential statement is \( 16^{\frac{1}{4}} = 4 \). The equivalent logarithmic statement is \( \log_{16} 4 = \frac{1}{2} \).
Example 3  Solve the equation $\log_6 36 = x$.

Rewriting the logarithmic statement as an exponential one produces

$6^x = 36$ or $6^x = 6^2$

Hence: $x = 2$.

Example 4  Solve the equation $\log_{0.5} x = -3$.

The answer follows straight from the definition.

$x = 0.5^{-3} = \left(\frac{1}{2}\right)^{-3}$

$= \left(2^{-1}\right)^{-3}$

$= 2^3 = 8$

Example 5  Solve the equation $\log_{x+1} 32 = 5$.

Here $(x+1)^5 = 32 = 2^5$.

or $x + 1 = 2$

Hence: $x = 1$

Because powers can be zero or negative, logarithms can be zero or negative. Note, however, that we can only take the logarithm of positive numbers. Also the base number is defined to be always positive.

The following are important relationships.

- $\log_x x = 1$
- $\log_x 1 = 0$

Example 6  Show $\log_x 1 = 0$.

If $\log_x 1 = 0$ then $x^0 = 1$ which is correct for any positive number $x$. 


Activity 4.1

1. Express these statements in an equivalent logarithmic form.

   a) \(3^x = 81\)  
   b) \(x^2 = y\)  
   c) \(\sqrt[4]{1.44} = 1.2\)

   d) \(2^{-3} = 0.125\)  
   e) \(2^2 = 4\)  
   f) \(w^2 = x\)

   g) \(10^1 = 10\)  
   h) \(1^y = 1\)  
   i) \(\sqrt{16} = 4\)

   j) \(8^{2/3} = 4\)  
   k) \(x^0 = 1\)  
   l) \(10^{-3} = 0.001\)

2. Restate in exponential form.

   a) \(\log_3 9 = 2\)  
   b) \(\log'1 = 0\)  
   c) \(\log_4 32 = 2.5\)

   d) \(\log_x N = y\)  
   e) \(\log_2 16 = 4\)  
   f) \(\log_4 2 = 0.5\)

   g) \(1 = \log_{12} 12\)  
   h) \(\log_3 1 = 0\)  
   i) \(4/3 = \log_{27} 81\)

3. Evaluate:

   a) \(\log_2 64\)  
   b) \(\log_4 64\)  
   c) \(\log_8 64\)

   d) \(\log_{64} 64\)  
   e) \(\log_3 9\)  
   f) \(\log_3 81\)

   g) \(\log_3\)  
   h) \(\log_{10} 1000\)  
   i) \(\log_{10} 0.000 0001\)

   j) \(\log_{49} 7\)  
   k) \(\log_1 1\)  
   l) \(\log_{0.25} 16\)

   m) \(\log_{25} 125\)  
   n) \(\log_2 2\sqrt{2}\)  
   o) \(\log_{81} 3\)

4. Solve the equations for x:

   a) \(\log_2 x = 4\)  
   b) \(\log_3 x = 3\)  
   c) \(\log_{16} x = -0.5\)

   d) \(\log_{10} x = -2\)  
   e) \(\log_x 7 = 1\)  
   f) \(\log_{0.1} x = -2\)

   g) \(\log_2 x = 0\)  
   h) \(\log_x 8 = 3\)  
   i) \(\log_x 27 = 3\)

   j) \(\log_6 36 = x\)  
   k) \(\log_6 x = 1\)  
   l) \(\log_x 1 = 0\)
4.2 Logarithm rules

You are familiar with the Index Rules. Because logarithms are in essence indices it should not be surprising that there is a corresponding set of rules for logarithms.

We shall only consider the three most important ones.

1. **Product rule**

Let \( p = \log_b x \) and \( q = \log_b y \) then \( x = b^p \) and \( y = b^q \)

On multiplying we obtain \( x \cdot y = b^p \times b^q = b^{p+q} \)

Rewriting this statement in logarithmic form we obtain:

\[
\log_b x \cdot y = p + q
\]

Hence: \( \log_b x \cdot y = \log_b x + \log_b y \)

2. **Quotient rule**

In similar manner we can derive that:

\[
\log_b \frac{x}{y} = \log_b x - \log_b y
\]

3. **Power rule**

Let \( r = \log_b x^n \) then \( x^n = b^r \)

On taking the \( n \)th root we derive that: \( x = \sqrt[n]{b^r} = (b^{\frac{r}{n}})^\frac{1}{n} \)

Hence: \( x = b^{\frac{r}{n}} \)

Rewriting this statement in logarithmic form we obtain: \( \frac{r}{n} = \log_b x \)

or \( r = n \log_b x \)

Hence: \( \log_b x^n = n \log_b x \)

**Example 1** Express \( 2 \log_4 3 - 3 \log_4 2 \) as a single logarithm.

\[
2\log_4 3 - 3\log_4 2 = \log_4 3^2 - \log_4 2^3
\]

\[
= \log_4 \left( \frac{3^2}{2^3} \right)
\]
Example 2  Solve for $x$ if $\log_3 (2x + 7) - \log_3 x = 2$.

If $\log_3 (2x + 7) - \log_3 x = 2$ then $\log_3 \left( \frac{2x + 7}{x} \right) = 2$.

In exponential form: $\left( \frac{2x + 7}{x} \right) = 3^2 = 9$.

Hence: $2x + 7 = 9x$ and thus $7x = 7$.

This leads to the solution $x = 1$.

Example 3  Express in terms of $y$ if $\log_{10} 9 = y$.

a) $\log_{10} 81$  b) $\log_{10} 3$  c) $\log_{10} 90$

a) $\log_{10} 81 = \log_{10} 9^2 = 2\log_{10} 9 = 2y$

b) $\log_{10} 3 = \log_{10} \sqrt{9} = \log_{10} 9^{\frac{1}{2}} = \frac{1}{2}\log_{10} 9 = \frac{1}{2}y$

c) $\log_{10} 90 = \log_{10} (9 \times 10) = \log_{10} 9 + \log_{10} 10 = y + 1$.

Example 4  Evaluate:

Let $x = 64^{\log_{10} 6}$ i.e $x = (4^3)^{\log_{10} 6} = 4^{3\log_{10} 6}$

Hence $x = 4^{3\log_{10} 6}$

Then from the definition of a logarithm: $\log_{4^3} x = \log_{4^3} 6^3$

and $x = 6^3 = 216$.

Activity 4.2

1. Simplify:

a) $\log_{10} 10^{-1}$  b) $\log_{4} 4^2$  c) $\log_{a} a^3$

d) $\log_{a} a^{p+q}$  e) $\log_{7} 7^7$  f) $\log_{a} \frac{1}{a}$

2. Find the value of:

a) $\log_{10} 2 + \log_{10} 5$  b) $\log_{4} 2 + \log_{4} 2$  c) $2\log_{10} 5 + \log_{10} 4$

d) $\log_{2} 16 + \log_{2} 8$  e) $\log_{3} 27 - \log_{3} 3$  f) $\log_{a} 2 + \log_{a} 5 - \log_{a} 10$
3. Given $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, evaluate without using a calculator.

   a) $\log_{10} 6$
   b) $\log_{10} 9$
   c) $\log_{10} 12$
   d) $\log_{10} \sqrt{3}$
   e) $\log_{10} 36$
   f) $\log_{10} 0.6$

4. Given $\log_b 2 = a$, express in terms of $a$.

   a) $\log_b 4$
   b) $\log_b 1024$
   c) $\log_b 0.5$
   d) $\log_b 0.125$
   e) $\log_b \sqrt{2}$
   f) $\log_b \frac{2}{\sqrt{2}}$

5. State the value of $x$:

   a) $\log_b 3 + \log_b 5 = \log_b x$
   b) $\log_b 8 - \log_b 2 = \log_b x$
   c) $\log_b x = 0.5 \log_b 16$
   d) $\log_b x = \frac{\log_b 27}{3}$

### 4.3 Special logarithms

There are two special types of logarithms which have special buttons on the calculator.

#### 1. The common logarithm

This logarithm has 10 as the base. Because it is used so frequently in calculations, the 10 is not written.

Thus $\log y$ means the same as $\log_{10} y$.

Using the definition of a logarithm we deduce that

If $x = \log y$ then $y = 10^x$.

**Example 1**  What is the value of $\log 1000$?

If $x = \log 1000$ then $10^x = 1000$

or

$10^x = 10^3$

Hence: $x = 3$.

The special button labelled log can be used to find values of common logarithms.
Example 2  
Obtain correct to four decimal places log 34.28.

Direct from the calculator, log 34.28 = 1.5350 (correct to 4 dp).

Example 3  
If \(10^x = 89.656\), what is \(x\)?

This is a logarithm problem in disguise. Rewriting the exponential statement in logarithmic form gives us

\[ x = \log 89.656 \]
\[ = 1.9526 \text{ (correct to 4 dp)} \]

Example 4  
Solve for \(x\): \(\log (2x + 1) = -1\)

In exponential format we obtain \(10^{-1} = 2x + 1\)

or \(0.1 = 2x + 1\)

or \(2x = (-0.9)\)

or \(x = \frac{(-0.9)}{2} = (-0.45)\)

2. The natural logarithm

The number \(e\) is approximately equal to 2.718 281 828 459 045 ……….. It is the limit value of the expression \(\left(1 + \frac{1}{n}\right)^n\) when \(n\) increases indefinitely.

Like \(\pi\), the number \(e\) is a transcendental number, ie a number that is irrational and not the root of an algebraic equation. The symbol \(e\) was first used by the Swiss mathematician Leonhard Euler (pronounced ‘Oiler’) who lived in the 18th Century, possibly inspired by the word ‘exponential’. Today, we regard naming the number as \(e\) more as a homage to Euler. Indeed, it is sometimes called Euler’s number but should not be confused with Euler’s constant which has a different value.

The number \(e\) has many special properties, some of which we shall encounter later in this Section. It occurs frequently in formulae dealing with natural phenomena and is extremely important in Calculus and Statistics.

When \(e\) is used as the base of a logarithm, the logarithm is written as \(\ln\). To avoid confusion, \(\ell n\) is frequently used.

Thus, \(\ln y\) means the same as \(\log_e y\).

Using the definition of a logarithm we deduce that:

\[ x = \ln y \text{ then } y = e^x \]

It is important to realise that:

- \(\ln e^x = x\)
- \(\ln 1 = 0\)
**Example 1** Evaluate \( \ln 5.78 \) correct to four decimal places.

Direct from the calculator, we obtain \( \ln 5.78 = 1.7544 \).

**Example 2** Evaluate \( e^{2.1} \) correct to four decimal places.

Using the ‘shift’ button and then the ln button, from the calculator, we obtain \( e^{2.1} = 8.1662 \).

**Example 3** Evaluate \( \log(\ln 5) \) correct to four decimal places.

First obtain \( \ln 5 \) and then take the common log of that number.

Thus \( \log(\ln 5) = \log (1.60943...) = 0.2067 \).

(Most calculators allow you to enter the whole expression as written, which avoids loss of accuracy by having to round too early.)

**Example 4** Solve the equation \( \frac{2.1x}{5} = e^2 \) to the nearest integer.

Evaluating \( e^2 \) first gives \( \frac{2.1x}{5} = 7.38905... \).

Then using the definition of a logarithm, the base here being \( e \) gives:

\[
\frac{2.1x}{5} \quad \text{or} \quad 1618.17799... = \frac{2.1x}{5}
\]

Then \( x = \frac{5}{2.1} \times 1618.17799... = 4045.44487... \)

Hence: \( x = 3853 \).

**Activity 4.3**

1. Use your calculator to obtain the following correct to four decimal places.

   a) \( \log 2 \)   b) \( \ln 5 \)   c) \( \log 24.7 \)
   d) \( \log e \)   e) \( \ln 10 \)   f) \( \log (2e + 1) \)
   g) \( \ln 5.87 \)   h) \( \frac{\log 35}{\log 18} \)   i) \( \frac{\ln 35}{\ln 18} \)
   j) \( \ln(\ln 2.1) \)   k) \( (\log 1.02)^{-1} \)   l) \( \log (3.98 \times 10^5) \)
2. Calculate correct to four decimal places.

   a) \(10^{2.07}\)  
   b) \(5e^{-2.7}\)  
   c) \(7.76e^{-5.9871}\)
   
   d) \(e^{2.71} + e^{2.98}\)  
   e) \(e^{\log 2.1}\)  
   f) \(10^{e^{-1}}\)
   
   g) \(\frac{1}{e}\)  
   h) \(\frac{\ln 2.4}{e^{2.4}}\)  
   i) \(\frac{\log e^{10}}{\ln 10^5}\)

3. Calculate \(x\) correct to four decimal places.

   a) \(\log x = 3.17\)  
   b) \(\ln x = 2.98\)  
   c) \(3\log x = e^{2.3}\)
   
   d) \(2 + 2.1 \log x = e\)  
   e) \(\ln \left(\frac{3.1}{x}\right) = e^{1.2}\)  
   f) \(\frac{\ln x}{\ln 3.1} = 2.5410\)
   
   g) \(\log(2x - 1) = 0.3\)  
   h) \(\ln(3x + 2) = 1.7\)  
   i) \(2\log(x - 1) = e^{1.4}\)

4.4 Change of base formula

It may happen that we need to calculate a logarithm with a base that is not 10 or \(e\). For example, how would we evaluate \(\log_5 34\)? Here the change of base formula comes in.

Consider \(\log_b y = x\). In exponential form we obtain \(b^x = y\).

Now let us take logarithms with base \(c\) from both sides. We obtain:

\[
\log_c b^x = \log_c y
\]

or

\[
x \log_c b = \log_c y
\]

and hence

\[
x = \frac{\log_c y}{\log_c b}
\]

But \(x = \log_c y\)

Therefore

\[
\log_c y = \frac{\log_c y}{\log_c b}
\]

Example 1 Evaluate \(\log_4 8\).

Changing to base 2: \(\log_4 8 = \frac{\log_2 8}{\log_2 4}\)

\[= \frac{\log_2 2^3}{\log_2 2^2}\]

\[= \frac{3\log_2 2}{2\log_2 2}\]

\[= 1.5\]
In practice we would like to change to a base 10 or e so that the calculator can be used. Thus the following are particularly useful:

\[
\log_b y = \frac{\log y}{\log b} \quad \text{and} \quad \log_b y = \frac{\ln y}{\ln b}
\]

**Example 2** Express \(\log_3 34\) in terms of a common logarithm.

We need to change from base 3 to base 10.

Changing to base 10: \(\log_3 34 = \frac{\log 34}{\log 3}\)

\[= 2.1 \log 34 \text{ using one decimal place}\]

**Example 3** Calculate correct to four decimal places \(\log_5 34\).

\[\log_5 34 = \frac{\log 34}{\log 5} = \frac{1.53147 \ldots}{0.69897 \ldots} = 2.1911\]

**Example 4** Evaluate \(\log_7 3152\) correct to three decimal places using natural logarithms.

\[\log_7 3152 = \frac{\ln 3152}{\ln 7} = \frac{8.05579 \ldots}{1.94591 \ldots} = 4.140\]

We finish this topic with a more difficult problem.

**Example 5** Given \(\log_5 x = 4 \log_5 5\), calculate the possible values of \(x\).

If \(\log_5 x = 4 \log_5 5\) then \(\log_5 x = \log_5 5^4\)

\[\text{or } \log_5 x = \frac{\log_5 5^4}{\log_5 x}\]

\[\text{or } \log_5 x = \frac{4\log_5 5}{\log_5 x}\]

This leads to \((\log_5 x)^2 = 4\)

From which \(\log_5 x = ± 2\)

Hence: \(x = 5^2 = 25\) or \(x = 5^{-2} = 0.04\)
Activity 4.4
1. Evaluate the following to four decimal places.
   a) $\log_3 2$  
b) $\log_7 5$  
c) $\log_6 24.7$
   d) $\log_5 e$  
e) $\log_{0.7} 1.7$  
f) $\log_3 21.7$
   g) $\log_3 2 + \log_3 3$  
h) $\frac{\log_5 35}{\log_3 18}$  
i) $\frac{\log_{0.3} 13.5}{\log_{0.3} e}$

2. Convert the following:
   a) $\ln 4.5$ to $\log_3$  
b) $\log 4.789$ to $\ln$  
c) $\log 2 5$ to $\log 3 5$

3. If $\log 2 = a$ and $\log 3 = b$, express $\frac{\log 27}{\log 18}$ in terms of $a$ and $b$.

4.5 Solving exponential equations
An exponential equation is one in which the unknown, usually $x$, appears as the exponent or index. A typical example is $3^x = 10$.

In this example, 10 is not a power of 3 and hence we cannot solve the equation by equating base numbers as we did previously. We shall use logarithms instead.

There are in fact two related methods for solving exponential equations using logarithms. You should be familiar with both.

We shall illustrate both processes.

1. The change of base method
Example 1 Solve $2^x = 9$ correct to two decimal places using the change of base method.

If $2^x = 9$ then using the definition of a logarithm: $x = \log_2 9$.

Changing to base 10 produces: $x = \frac{\log 9}{\log 2}$

$$= \frac{0.9542...}{0.3010...} = 3.17$$
2. The logarithm law method

Example 2  Solve $2^x = 9$ correct to two decimal paces using the logarithm law method.

This method involves taking (common) logarithms from both sides.

Thus the equation becomes:  \[ \log 2^x = \log 9. \]

Using the third logarithm rule:  \[ x \log 2 = \log 9 \]

or  \[ x = \frac{\log 9}{\log 2} = 3.17 \]

As with other equations, we should always check whether the answer makes sense. Because we know that $2^3 = 8$, we expected an answer just slightly more than 3.

It is also important to remember that before any of the two methods can be employed, there should be no coefficient attached to the base number. Thus if an equation such as $3.2 \times 2^x = 9$ needs to be solved, it must first be written as $2^x = \frac{9}{3.2}$.

The first method is generally easier if natural logarithms are involved. Consider this example.

Example 3  Solve $4e^{2t} = 9$ correct to two decimal places.

If $4e^{2t} = 9$, then $e^{2t} = \frac{9}{4} = 2.25$.

Direct from the definition we obtain:  \[ 2t = \ln 2.25 \]

or  \[ t = \frac{\ln 2.25}{2} \]

and hence:  \[ t = 0.41 \]

In this last example we could have taken natural logarithms from both sides (after first dividing by 4) to obtain $\ln e^{2t} = \ln 2.25$. Then using the fact that $\ln e^x = x$ we would get to $2t = \ln 2.25$ from which we would continue as before.

The second method is longer but is easier to use especially when the index expression is more complicated and does not involve $e$.

Let us illustrate the second method with two more examples. You will notice that we refrain from using the calculator too early to obtain values of logarithms. Doing this ensures that rounding errors do not affect our final answer.

Example 4  Solve $(0.5)^{x + 2} = 4.6$ correct to three decimal places.

Taking logarithms from both sides:  \[ \log (0.5)^{x + 2} = \log 4.6 \]

or  \[ (x + 2)\log 0.5 = \log 4.6 \]

or  \[ x\log 0.5 + 2\log 0.5 = \log 4.6 \]

or  \[ x\log 0.5 = \log 4.6 - 2\log 0.5 \]

or  \[ x = \frac{\log 4.6 - 2\log 0.5}{\log 0.5} \]

Hence:  \[ x = -4.202 \]
Example 5  Solve $3^{x-1} = 2^{x+1}$ correct to three decimal places.

Taking logarithms from both sides: $\log 3^{x-1} = \log 2^{x+1}$

$(x - 1)\log 3 = (x + 1)\log 2$

or $x \log 3 - \log 3 = x \log 2 + \log 2$

or $x \log 3 - x \log 2 = \log 2 + \log 3$

or $x = \frac{\log 2 + \log 3}{\log 3 - \log 2}$

Hence: $x = 4.419$

Activity 4.5

If possible, solve the following exponential equations correct to four decimal places.

1. $2^x = 15$
2. $3^x = 89$
3. $(3.2)^x = 2.5$
4. $e^x = 42$
5. $e^{2x} = 52.4$
6. $5 \times 7^x = 81$
7. $10^x = 1.2344$
8. $e^{x+1} = 12$
9. $2^{2x} = 3^2$
10. $2^{2x} = 3^x$
11. $3^x + 24.9 = 71.9$
12. $3(2.1)^x = 89.987$
13. $23.22 = 65.9e^x$
14. $(2.1)^{x+1} = (3.1)^{x-1}$
15. $8^{2x} = 3^{x+2}$
16. $11^{x-3} = 2^{2x-2}$
17. $e^{2x} = (2e)^x$
18. $2^x = 2.1 \times 3^y$
19. $2^{-x} = 12.7$
20. $9.01 + 2^{-x} = 34.55$
4.6 Formulae involving logarithms

Let us return to the formula \( S = P(1 + r)^t \) which is used in financial mathematics to calculate how long \( t \) a principal sum \( P \) needs to be invested at a given percentage \( r \) to produce an accumulated sum \( S \).

This formula is essentially an exponential equation and if we wanted to obtain a formula with \( t \) as the subject, we would need to use logarithms. The process is outlined in the following example.

**Example 1**
Make \( t \) the subject in the formula: \( S = P(1 + r)^t \)

Dividing by \( P \) first:
\[
\frac{S}{P} = (1 + r)^t
\]
Taking logarithms:
\[
\log\left(\frac{S}{P}\right) = \log(1 + r)^t
\]
Using logarithms rules:
\[
\log S - \log P = t \log (1 + r)
\]
Dividing by \( \log(1 + r) \):
\[
t = \frac{\log S - \log P}{\log (1 + r)}
\]

As you will have noticed, transposition problems involving logarithms are quite difficult. Let us study another important example.

**Example 2**
Make \( t \) the subject of \( A = A_0 e^{kt} \)

Dividing first by \( A_0 \):
\[
\frac{A}{A_0} = e^{kt}
\]
Using the definition of \( \ln \):
\[
\ln\left(\frac{A}{A_0}\right) = kt
\]
Using the logarithm rule:
\[
\ln A - \ln A_0 = kt
\]
Dividing by \( k \):
\[
t = \frac{\ln A - \ln A_0}{k}
\]
Activity 4.6
For questions 1–6, make x the subject of the formula.

1. \( a = b^x \)  
2. \( w = y + 10^x \)  
3. \( \log (5 - x) = z \)  
4. \( z = ke^x \)  
5. \( \log \left( \frac{k}{x} \right) = z \)  
6. \( \log x = \log y \)

7. The characteristic impedance of a parallel conductor (twin head), where dry air separates the conductors, is given by \( Z = 276 \log \left( \frac{2D}{d} \right) \) where \( D \) is the separation of the conductors and \( d \) is the diameter of the conductor. Transpose this formula to make \( d \) the subject.

4.7 Graph of the logarithmic function
The logarithmic function is defined as \( f(x) = \log_b x \) where \( b > 0 \) and \( b \neq 1 \).

Let us consider two examples of graph of this function.

Example 1  Draw the logarithmic graph defined by the rule \( y = \log_2 x \).

\[
\begin{array}{c|cccccc}
 x & 0.5 & 1 & 2 & 4 & 8 \\
 y & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

If we compare this graph with the graph of \( y = 2^x \), we notice that the graphs are reflections of each other in the line \( y = x \). In mathematical terms, the exponential and logarithmic functions are each other’s inverse.
Example 2  Graph \( y = 2 \log_4 x \) and \( y = 1 - \log_2 x \) and use your graph to solve \( 2 \log_4 x + \log_2 x = 1 \).

\[ 2 \log_4 x + \log_2 x = 1 \] can be rewritten as \( 2 \log_4 x = 1 - \log_2 x \)

Thus to solve this equation graphically is to find the \( x \) coordinate of the intersection of the graphs of \( y = 2 \log_4 x \) and \( y = 1 - \log_2 x \).

The intersection occurs at approximately \( x = 1.4 \). Hence \( x \approx 1.4 \)

(You may wish to verify that the exact value is \( \sqrt{2} \).)
Activity 4.7

Draw a sketch of the following functions, indicating important features.

1. \( y = \log x \)  
2. \( y = \ln x \)  
3. \( y = \log_5 x \)  
4. \( y = -\log x \)

In questions 5 and 6, draw both graphs on the same set of axes and find their intersection point if it exists.

5. \( y = \log x + 1 \) and \( y = \log(x + 1) \)  
6. \( y = -\ln x \) and \( y = \frac{1}{\ln x} \)

7. On the same axes draw \( y = 10^x \) and \( y = \log x \).

Describe how the graphs are related.

8. Use your calculator to draw graphs of \( y = \log 5x \) and \( y = x^2 - 1 \) and find, accurately to one decimal place, the coordinates of the intersection point(s).

4.8 Exponential growth

You may have heard the expression that something such as, the population of a country has, ‘grown exponentially’. It is in this type of problem that logarithms are used extensively.

Exponential growth is characterised by the fact that at any particular time the rate of growth is proportional to the amount that is present. This implies that for any exponentially growing quantity, the larger the quantity gets, the faster it grows.

Typical examples of exponential growth are heating of a fluid and charging of a battery. A fluid can also cool down or a battery can discharge. In these cases we speak of exponential decay. This occurs when, at any particular time, the rate of decay is proportional to the amount present.

As we shall see later, exponential decay can be treated in exactly the same way as exponential growth. For this reason we shall, at least for the time being, concentrate on exponential growth problems.

A requirement for exponential growth is that the process occurs continuously over time. Many phenomena that grow can be approximated by the mathematics of exponential growth but are, strictly speaking, not exponential growth problems. Compound interest and bacteria growth are typical examples. These can best be called ‘stepwise exponential growth’ problems. Let us investigate this matter in more detail.
Example 1  A sales representative accepts a job at $50 per hour and is promised that her salary will rise by 25% each year. After working four years with the company what will be her salary at the start of her fifth year?

From the table we observe that the sales representative will earn $122.07 per hour at the start of her fifth year.

<table>
<thead>
<tr>
<th>Start Year</th>
<th>Salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$50</td>
</tr>
<tr>
<td>2</td>
<td>$50 + 25% of $50 = $62.50</td>
</tr>
<tr>
<td>3</td>
<td>$62.50 + 25% of $62.50 = $78.12</td>
</tr>
<tr>
<td>4</td>
<td>$78.12 + 25% of $78.12 = $97.66</td>
</tr>
<tr>
<td>5</td>
<td>$97.66 + 25% of $97.66 = $122.07</td>
</tr>
</tbody>
</table>

This final amount could have been worked out quickly by calculating the value of $50 \times (1.25)^4$ where the 1.25 comes from expressing $100\% + 25\% = 125\%$ as a decimal. Every year the hourly amount is multiplied by this factor. Notice that her yearly increases become progressively bigger because they are calculated on a progressively bigger salary.

If, in the last example, we had graphed the sales representative’s salary we would notice that the steps become progressively bigger. When we connect her starting salary of each year, we clearly see the exponential nature of her salary progression. Note, however, that the smooth curve does not represent her salary situation. In the middle of her third year, for example, she does not earn $50 \times (1.25)^{2.5} = $87.35 as suggested by the curve but receives her starting salary for that year which is $78.12.
You may think that the smooth exponential curve represents exponential growth but that is not the case. Remember that growth has to occur on a continuous basis at any time, not just at the beginning of a year. Even if the sales representative gets monthly increases at the rate of 25% 12 or for that matter weekly increases of 25% ÷ 52, the growth would still not be of a continuous exponential nature but would, however, approximate it.

The difference between getting a salary increase on a yearly basis or on a continuous basis is illustrated in the graph below. In fact, had she received salary increases according to the continuous exponential growth model, she would be on $135.91 instead of $122.07 per hour at the start of her fifth year. Although not illustrated, salary increases on a monthly basis would have given her $134.52 per hour while salary increases on a weekly basis would have seen her salary rise to $135.59 per hour.

Salary increases do not occur on a continuous basis. The heating of a metal or the decay of radioactivity, on the other hand, do continuously change exponentially over time.

For these processes we use a model, called the exponential growth model, that can be derived from Calculus. The formula for this model is:

$$A = A_o e^{kt}$$

Where:  
- $A =$ the Final Amount  
- $k =$ growth proportion per time unit  
- $A_o =$ the Initial Amount  
- $t =$ the number of time units

Let us illustrate the use of this formula with a few examples.
Example 2  Water is heated in a kettle at the rate of 40% per minute. What will be its temperature after 3 minutes if the water was initially at room temperature of 20 °C? Use one decimal place accuracy.

Here \( A_o = 20 \), \( k = 40\% = 0.4 \) and \( t = 3 \)

Then \( A = 20e^{(0.4 \times 3)} = 66.4 \) °C

Example 3  Water is heated in a kettle at the rate of 40% per minute. How long, to the nearest second, will it take to reach boiling point if the water was initially at room temperature of 20 °C?

Here \( A = 100 \), \( A_o = 20 \) and \( k = 40\% = 0.4 \).

Then \( 100 = 20e^{(0.4 \times t)} \) or \( \frac{100}{20} = e^{(0.4 \times t)} \)

or \( 5 = e^{(0.4 \times t)} \)

Using the definition of a logarithm, we obtain

\[ 0.4 \times t = \ln 5 \]

or \( t = \frac{\ln 5}{0.4} = 4.02349 \ldots \) min

= 4 minutes and 2 seconds

Example 4  A faster kettle is used, one that boils the water from room temperature of 20 °C to boiling point in 3 minutes. At what rate, correct to the nearest one tenth of a percent, does this kettle boil water?

Here \( A = 100 \), \( A_o = 20 \) and \( t = 3 \).

Then \( 100 = 20e^{(r \times 3)} \) or \( \frac{100}{20} = e^{(r \times 3)} \)

or \( 5 = e^{(r \times 3)} \)

Using the definition of a logarithm, we obtain

\[ r \times 3 = \ln 5 \]

or \( r = \frac{\ln 5}{3} = 0.536479\ldots \)

= 53.6% per minute
As mentioned earlier, exponential decay can be treated in exactly the same manner. Consider this example.

**Example 5** A battery loses 50% of its charge over a period of 20 days. How much is this per day? Assume that a fully loaded battery has a charge of 100 units.

Let the fully loaded battery have 100 units.

Here \( A = 50 \), \( A_0 = 100 \) and \( t = 20 \).

Then \( 50 = 100e^{(k \times 20)} \) or \( \frac{50}{100} = e^{(k \times 20)} \)

or \( 0.5 = e^{(k \times 20)} \)

Using the definition of a logarithm, we obtain

\[
k \times 20 = \ln 0.5
\]

or \( k = \frac{\ln 0.5}{20} = -0.034657... \)

= 3.47% per day using 2 dp

In this last example, we noticed that the answer was negative, indicating negative growth or decay. When given a decay rate, you have to insert the negative sign into the equation.

In this example, we actually calculate the half life of the battery, i.e., the time it takes for half the amount to disappear. This concept is frequently used in conjunction with radioactivity.

We finish with a difficult example in which first \( k \) and then \( A_0 \) need to be calculated before the question can be answered. These values have deliberately been left 'uncalculated' to avoid loss of accuracy. You will notice that doing this enables us to cancel using index rules.

**Example 6** A radioactive substance has a half life of 2010 years. If after 500 years 82 mg was present, how much, correct to 2 decimal places, will be present after 4000 years?

We first calculate \( k \).

Here \( A = 41 \), \( A_0 = 82 \) and \( t = 2010 \).

Then \( 41 = 82e^{(k \times 2010)} \) or \( e^{(k \times 2010)} \)

or \( 0.5 = e^{(k \times 2010)} \)

Using the definition of a logarithm, we obtain

\[
k \times 2010 = \ln 0.5
\]

or \( k = \frac{\ln 0.5}{2010} \)
Now let's calculate the initial amount.

\[ A = 82, \ k = \frac{\ln 0.5}{2010} \quad \text{and} \quad t = 500 \]

Then \( 82 = A_o e^{\left(\frac{\ln 0.5}{2010}\right) t} \) or \( A_o = \frac{82}{e^{\left(\frac{\ln 0.5}{2010}\right) t}} \)

Finally, we can calculate the amount after 4000 years.

\[ A_o = \frac{82}{e^{\left(\frac{\ln 0.5}{2010}\right) t}}, \ k = \frac{\ln 0.5}{2010} \quad \text{and} \quad t = 4000. \]

\[ A = \frac{82}{e^{\left(\frac{\ln 0.5}{2010}\right) t}} e^{\left(\frac{\ln 0.5}{2010}\right) (4000)} = 24.53 \text{ g} \]

**Activity 4.8**

1. If \( V = Ae^{kt} \), find:
   a) \( V \) if \( A = 5, \ k = 0.07 \) and \( t = 6 \)
   b) \( t \) if \( A = 2000, \ k = 0.13 \) and \( V = 3600 \)

2. If \( N = 600e^{-0.015t} \), find:
   a) \( N \) when \( t = 25 \)
   b) \( t \) when \( N = 250 \)

3. The rate constant \( K \) for a chemical reaction is given by the equation:

\[ \ln K = \frac{-E_a}{RT} + B \]  

where \( E_a \) is the activation energy and \( B \) is a constant.

Find \( K \) if \( T = 40 \ ^\circ \text{C}, B = 1.90, \ R = 5000 \) and \( E_a = 10^6 \text{ calories} \).

4. The growth rate per hour of a population of bacteria is 9% of the population. Initially the population is 4000. What is the population after 6 hours? (Give your answer correct to the nearest 100 bacteria.)

5. A colony of bacteria grows at the rate of 7% per hour. If originally there are 200 bacteria present, how many will there be after 2 hours?

6. The population of a small town grows from 9000 to 11\,000 in ten years.
   a) Find the annual growth rate to the nearest percent, assuming that it is proportional to the population.
   b) Calculate the population of the town 25 years after the initial count.
7. The population of a town increases from 10 400 at the rate of 11% each year. How many people will be in the town 18 months from now?

8. A radioactive substance is known to decompose at a rate proportional to the mass present at any time. If a mass of 80 g decomposes to 75 g after 5 years, find:
   a) the mass present after 20 years (correct to the nearest gram)
   b) the time taken for the mass to decay to 30 g (correct to the nearest year).

9. The level of radioactivity of a substance decreases by 20% each year. If originally there was 240 mg present, how much will there be 30 years from now?

10. The temperature of a piece of coal decreases exponentially at 8% per minute until room temperature is reached. Initially the temperature of the coal was 100 °C. What will the temperature be after 4 minutes?

11. The level of a certain radioactive substance decreases exponentially by 20% every year. What percentage of radioactivity remains at the end of the fifth year?

12. The population of a city was 800 000 last year. In how many years from now do you expect the population to exceed 1 000 000 if the growth rate is 10% per year?

13. The number of bacteria doubles every three days. There are 100 000 now. How many are there after 4 days?

14. A certain population is increasing at a rate proportional to the population present and will double in 30 years. How many years will it take for the population to triple?

15. The half-life of radium is 1 690 years. If 50 mg is present now, how much will be present 100 years from now?

16. The oil exports of a certain Middle East country are increasing according to the law of natural growth. If exports increased 50% from 1990 to 1998, how much can they expect to increase by from 1998 to 2004?

17. Water drains from a storage tank at a rate proportional to the volume present. Initially the tank holds 1000 litres but after 40 minutes only 800 litres remain.
   a) How much water is in the tank after 1 hour?
   b) How long will it take to reach the last litre of water in the tank?
18. Electrical failure causes a train to slow down at a rate proportional to its velocity. A train is travelling at 60 km/h when the power fails. Twenty seconds later its speed is 50 km/h. What will be the train’s speed one minute after the power failure?

19. The law of healing of a skin wound is given by \( A = Be^{-0.15t} \) where \( A \) is the area in cm\(^2\) of the unhealed portion of the skin wound \( t \) days after it was sustained and \( B \) is the area in cm\(^2\) of the original skin wound.

A football player suffers a nasty leg wound in a match and is told that he cannot play until at least 80% of the wound is healed. His next match is in ten days. Is he able to play?

20. If $10,000 is invested at 4% p.a. compound interest what will this amount become after one year if interest is calculated continuously?

### 4.9 The power function in linear form

Sometimes experimental data is collected and it is suspected or discovered that the relationship between two variables is of a form that involves an exponent. How can we determine the mathematical relationship or law connecting the two variables?

We shall investigate two types of relationship, the **power function** type and the **exponential function** type, and discover that logarithms can be used.

Let us first clearly state the difference between the two types. If ‘\( A \)’ and ‘\( n \)’ are constant then the power and exponential functions are defined as in this table.

<table>
<thead>
<tr>
<th>Power Function</th>
<th>Exponential Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = Ax^n )</td>
<td>( y = Ae^x )</td>
</tr>
</tbody>
</table>

**Example 1** Classify as a power or exponential function.

a) \( y = 3.1 \times (2.4)^x \)  
b) \( y = e^x \)  
c) \( A = B^t \)

d) \( P = 3z^{7.1} \)  
e) \( y = \frac{5}{x^2} \)  
f) \( y = 2.3e^{2.4x} \)

The powers are known in c), d) and e) and hence they are power functions. In a), b) and f) the power appears as a variable which is characteristic of exponential functions.
In this topic we examine the power function and show how this function may be transformed into a linear form. In the next topic we deal with the exponential function.

Consider: \( y = Ax^n \)

Taking logarithms of both sides:

\[
\log y = \log(Ax^n) = \log A + \log x^n = \log A + n \log x
\]

This can be rewritten as: \( \log y = n \log x + \log A \)

If we let \( Y = \log y, X = \log x \) and \( b = \log A \), then this equation becomes:

\[
Y = nX + b
\]

This is the equation of a straight line with gradient \( m = n \) and \( y \) intercept \( b = \log A \).

This means that when we plot \( X \) against \( Y \) – that is, \( \log x \) against \( \log y \) – we should see a straight line result for which we should be able to determine the gradient and \( y \) intercept and consequently the values of \( n \) and \( A \).

Let us illustrate this process by means of an example.

**Example 2**

The tractive force \( F \) of an electromagnet for different values of the induction \( B \) is given by the following set of values:

<table>
<thead>
<tr>
<th>( B )</th>
<th>1 000</th>
<th>3 000</th>
<th>5 000</th>
<th>7 000</th>
<th>9 000</th>
<th>10 000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>41</td>
<td>365</td>
<td>1 014</td>
<td>1 987</td>
<td>3 286</td>
<td>4 056</td>
</tr>
</tbody>
</table>

If the law connecting \( F \) and \( B \) can be expressed as \( F = k B^n \)
find the values of \( k \) and \( n \).

Taking logarithms of both sides of the equation:

\[
\log F = \log k + \log B^n = \log k + n \log B
\]

ie \( \log F = n \log B + \log k \)

The equation has now been reduced to the linear form \( Y = mX + b \)

Where: \( \log F \) is represented by \( Y \)

\( \log B \) is represented by \( X \)

\( n \) is represented by \( m \)

\( \log k \) is represented by \( b \)

To find the values of \( k \) and \( n \) we plot the graph of \( \log F \) (vertical axis) versus \( \log B \) (horizontal axis) and draw the line that best fits the data.
Below are tabulated the values of \( F \) and \( B \) with their respective logarithms:

<table>
<thead>
<tr>
<th>( B )</th>
<th>( \log B )</th>
<th>( F )</th>
<th>( \log F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 000</td>
<td>3.000</td>
<td>41</td>
<td>1.613</td>
</tr>
<tr>
<td>3 000</td>
<td>3.477</td>
<td>365</td>
<td>2.562</td>
</tr>
<tr>
<td>5 000</td>
<td>3.699</td>
<td>1 014</td>
<td>3.006</td>
</tr>
<tr>
<td>7 000</td>
<td>3.845</td>
<td>1 987</td>
<td>3.298</td>
</tr>
<tr>
<td>9 000</td>
<td>3.954</td>
<td>3 286</td>
<td>3.516</td>
</tr>
<tr>
<td>10 000</td>
<td>4.000</td>
<td>4 056</td>
<td>3.608</td>
</tr>
</tbody>
</table>

The following graph results.

Now from the graph we see that the y intercept of the line of best fit is \(-4.4\). This means that \( \log k \approx -4.4 \).

Hence: \( k = 10^{-4.4} \approx 0.000\ 04 \)

Remember that the gradient measures the ratio \( \text{rise: run} \) or \( \frac{\text{rise}}{\text{run}} \).

Both the \( \text{rise} \) and \( \text{run} \) can be obtained graphically by taking two convenient points on the line. It can be seen that the \( \text{rise} \) is approximately 6 for a \( \text{run} \) of 3.

Hence: \( n \approx \frac{6}{3} = 2 \).

We thus find that the law connecting \( F \) and \( B \) is: \( F = 0.000\ 04B^2 \)

A check should always be made to ensure that the constants have been correctly evaluated. To do this we take a random point and check whether the point obeys the rule found.
Example 3  In the previous example, check that the point (3000, 365) satisfies the law \( F = 0.000\, 04B^2 \)

If we substitute \( B = 3000 \) we obtain \( F = 0.000\, 04 \times 3000^2 = 360 \) which is close to 365. (The difference could well be due to rounding or the fact that not all points lie exactly on the line.) Hence the law is verified.

There is another graphical way in which coefficients such as \( n \) and \( k \) in the last example could be obtained, and that is by means of special graph paper called logarithmic graph paper. Note that in the last example we plotted a log value against a log value. This special graph paper has logarithm scales on both axes so there is no need to calculate the logarithms.

Logarithmic graph paper is not always readily available. Also, graphical methods generally do not provide accurate results. For this reason, it may be better to calculate the coefficients using algebra.

The best procedure for this is to take two points lying on or very near the line and not too close together, substitute their coordinates into the linear equation involving the logarithms and then solve the two equations simultaneously.

Example 4  In the previous example, use the points (1000, 41) and (10,000, 4056) to find the values of \( n \) and \( k \).

Previously we obtained \( \log F = n \log B + \log k \)

Substituting:

\[
\log 4056 = n \log 10000 + \log k \\
\log 41 = n \log 1000 + \log k
\]

Evaluating the logarithms:

\[
3.608 = 4n + \log k \\
−1.613 = 3n + \log k \\
1.995 = n
\]

Hence \( n = 1.995 \approx 2 \)

Then \( 3.608 = 4(1.995) + \log k \)

or \( \log k = 3.608 − 7.98 = −4.372 \)

Hence \( k = 10^{-4.372} = 0.000\, 04246… \approx 0.000\, 04 \)

We thus find that the law connecting \( F \) and \( B \) is \( F = 0.000\, 04B^2 \)

A graphics calculator or scientific calculator can also be used. In fact using a calculator we obtain \( k = 0.000\, 042157… \) and \( n = 1.99563088… \)

These values are a kind of average of all possible \( n \) and \( k \) values that we could have calculated based on taking different sets of two points. Rather than relying solely on your calculator you should use it to verify your results.

Yet another method of obtaining the linear form of a power function is to take two appropriate points from the graph and use these two points to obtain the gradient and y intercept.
This method is illustrated in the following example.

**Example 5**  The time, \( t \) seconds, that it took for water to flow through a triangular notch, under a head of \( h \) metres, until the same quantity was in each case discharged, was found by experiment to be as in the table. Find the law connecting \( t \) and \( h \) assuming it is of the form \( t = ah^n \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>0.043</th>
<th>0.060</th>
<th>0.077</th>
<th>0.094</th>
<th>0.100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>1260</td>
<td>450</td>
<td>275</td>
<td>170</td>
<td>135</td>
</tr>
</tbody>
</table>

Taking logarithms we get:

\[
\begin{align*}
X &= \log h \\
&= -1.3665 \\
&= -1.2218 \\
&= -1.1135 \\
&= -1.0269 \\
&= -1.0000
\end{align*}
\]

\[
\begin{align*}
Y &= \log t \\
&= 3.1004 \\
&= 2.6532 \\
&= 2.4393 \\
&= 2.2304 \\
&= 2.1303
\end{align*}
\]

The following graph results.

![Graph showing log t vs log h]

We establish that the points \((-1.3665, 3.1004)\) and \((-1.0000, 2.1303)\) are not ‘outliers’ and use these points to obtain the equation of the line using \( Y - Y_1 = m(X - X_1) \) with \( m = \frac{Y_2 - Y_1}{X_2 - X_1} \).

In combination, this produces:

\[
Y - 3.1004 = \frac{2.1303 - 3.1004}{-1.0000 - (-1.3665)} (X - (-1.3665))
\]

or \( Y - 3.1004 = -2.6469 (X + 1.3665) \)

or \( Y = -2.6469X - 0.5166 \)

Then \( n = -2.6469 \) and \( a = 10^{-0.5166} = 0.3044 \)

Thus the required law (using two decimal places) is \( t = 0.30h^{-2.65} \)

Check: When \( h = 0.077 \) then \( t = 0.30 (0.077)^{-2.65} \approx 275 \)
Activity 4.9

1. In an experiment to find the luminosity \( l \) of a lamp for varying voltage \( V \), the following measurements were made:

<table>
<thead>
<tr>
<th>( V ) (volts)</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
<th>160</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l ) (lux)</td>
<td>10</td>
<td>31.6</td>
<td>88</td>
<td>184</td>
<td>322</td>
<td>580</td>
</tr>
</tbody>
</table>

Show that the law is of the type \( l = aV^n \) where \( a \) and \( n \) are constants, and find the values of \( a \) and \( n \).

2. The following table gives the pressure \( p \) corresponding to volumes \( v \). Plot \( \log p \) against \( \log v \) and use the graph to find \( p \) in terms of \( v \).

<table>
<thead>
<tr>
<th>( v ) (m(^3))</th>
<th>26.43</th>
<th>22.40</th>
<th>19.08</th>
<th>16.32</th>
<th>14.04</th>
<th>12.12</th>
<th>10.50</th>
<th>9.15</th>
<th>8.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p ) (Pa)</td>
<td>14.70</td>
<td>17.53</td>
<td>20.80</td>
<td>25.54</td>
<td>28.80</td>
<td>33.70</td>
<td>39.25</td>
<td>45.50</td>
<td>52.50</td>
</tr>
</tbody>
</table>

3. The following values of \( H \) and \( Q \) are connected by a law of the type \( Q = aH^n \), where \( a \) and \( n \) are constants.

<table>
<thead>
<tr>
<th>( H )</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.2</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q )</td>
<td>6.087</td>
<td>6.751</td>
<td>7.316</td>
<td>7.571</td>
<td>7.927</td>
<td>8.467</td>
</tr>
</tbody>
</table>

Plot suitable quantities to obtain a straight-line graph and from it determine the values of \( a \) and \( n \).

4. The following table shows the pressure \( p \) and volume \( v \) of a quantity of air when subjected to moderately rapid compression.

<table>
<thead>
<tr>
<th>( p ) (Pa)</th>
<th>15</th>
<th>30</th>
<th>60</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v ) (m(^3))</td>
<td>10</td>
<td>5.61</td>
<td>3.15</td>
<td>2.06</td>
<td>15</td>
<td>0.65</td>
</tr>
</tbody>
</table>

Show that \( p \) and \( v \) are connected by a law of the form \( pv^n = k \), and find the law.
4.10 The exponential function in linear form

In the previous topic we examined how the power function can be transformed into linear form. In this topic we look at a similar process involving the exponential function $y = A e^{kx}$.

Because the exponential function has such widespread use in exponential growth and decay problems, rather than examining $y = A e^{kx}$, we will look at a special form of this function involving $e$, sometimes called the exponential function, i.e. $y = ae^{kx}$.

Calculators also display results in this format.

Let $y = ae^{kx}$ then taking natural logarithms we obtain:

$$
\ln y = \ln (ae^{kx}) = \ln a + \ln e^{kx} = \ln a + kx \ln e
$$

$$
\ln y = kx + \ln a \quad as \quad \ln e = 1
$$

If we let $Y = \ln y$ and $b = \ln a$, then this equation becomes:

$$
Y = mx + b
$$

This is the equation of a straight line with gradient $m = k$ and y intercept $b = \ln a$.

This means that when we plot $x$ against $Y$, i.e. $x$ against $\ln y$ we should see a straight-line result for which we should be able to determine the gradient and y intercept and consequently the values of $k$ and $a$.

As with the power function, there is another graphical way in which coefficients such as $k$ and $a$ can be obtained and that is by means of log-linear paper. This is because we can plot an actual value against the logarithm of another. It can be shown that using this type of paper rather than ln-linear paper does not change the values obtained.

In any case, this type of paper is not always available and we shall not use this method. Here we shall plot all points and use two points, not too close together and lying on or close to the line of best fit, to obtain the gradient and y intercept.

Example 1  Show the following values of $x$ and $y$ are connected by a law of the form $y = ae^{kx}$, and hence find the constants $a$ and $k$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>30.3</td>
<td>47.5</td>
<td>74.5</td>
<td>116.7</td>
<td>182.8</td>
<td>287.1</td>
<td>449.8</td>
</tr>
</tbody>
</table>

Changing $y$ to $\ln y$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = \ln y$</td>
<td>3.4111</td>
<td>3.8607</td>
<td>4.3108</td>
<td>4.7596</td>
<td>5.2084</td>
<td>5.6598</td>
<td>6.1088</td>
</tr>
</tbody>
</table>
Using (2, 3.4111) and (5, 6.1088) and \( Y - Y_1 = m(x - x_1) \)

with \( m = \frac{Y_2 - Y_1}{x_2 - x_1} \) produces:

\[
Y - 3.4111 = \frac{6.1088 - 3.4111}{5 - 2}(x - 2)
\]

or \( Y - 3.4111 = 0.8992(x - 2) \)

or \( Y = 0.8992x + 1.6113 \)

The values of the gradient and y intercept agree with the graph from which we could have obtained them.

Now \( m = k \) and the y intercept \( b = \ln a \) or \( a = e^b \).

Hence \( k = 0.8992 \) and \( a = e^{1.6113} = 5.0093 \).

Thus the required law (using two decimal places) is \( y = 5.01e^{0.90x} \)

Check: When \( x = 3 \) then \( y = 5.01e^{0.90 \times 3} \approx 74.5 \)
Activity 4.10

1. The tension $T$ in a rope wrapped around a rough curved block is found at points in the rope defined by an angle $\theta$. The corresponding values are:

<table>
<thead>
<tr>
<th>$\theta$ (radians)</th>
<th>0.2</th>
<th>0.4</th>
<th>0.7</th>
<th>0.9</th>
<th>1</th>
<th>1.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (newtons)</td>
<td>10.29</td>
<td>10.58</td>
<td>11.05</td>
<td>11.34</td>
<td>11.50</td>
<td>12</td>
</tr>
</tbody>
</table>

Test these values for a law of the type $T = Ae^{\mu \theta}$, and determine the most suitable values for the constants $A$ and $\mu$.

2. The following tables gives the amplitude ($A$) cm, of the damped vibrations of a pendulum swinging in a viscous medium after ($t$) seconds:

<table>
<thead>
<tr>
<th>$t$ (seconds)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$ (cm)</td>
<td>10</td>
<td>4.97</td>
<td>2.47</td>
<td>1.22</td>
<td>0.61</td>
<td>0.30</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Show that $A$ and $t$ are connected by a law of the form $A = ae^{kt}$, and find the law.

3. $Y$ is the coefficient of viscosity of glycerine at various temperatures $\theta$ °C. Find a formula of the type $Y = ae^{k \theta}$ by drawing a straight-line graph.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>46</td>
<td>21</td>
<td>8.5</td>
<td>3.5</td>
</tr>
</tbody>
</table>

4.11 Practical logarithm problems

Logarithms have many practical applications. We already saw a financial application. We also looked at growth and decay problems. There are many other fields of study, such as chemistry and electronic engineering, in which logarithms are used extensively. Logarithms are also extremely important in calculus.

Let us look at a couple of applications.

**Example 1** Use the formula $t = \frac{\log S - \log P}{\log (1 + r)}$ to calculate how long $1000 needs to be invested at 5.2% compound interest per year to accumulate to 1500.

With $P = 1000$, $S = 1500$ and $r = 0.052$,

$$t = \frac{\log 1500 - \log 1000}{\log (1 + 0.052)} = 7.9984...$$

Hence it takes 8 years.
Example 2 In 1935 Charles Richter defined the magnitude of an earthquake to be:

\[ M = \log \left( \frac{I}{S} \right) \]

where:

- \( I \) is the intensity of the earthquake and \( S \) is the intensity of a standard earthquake.

The 1985 Mexico City earthquake had a magnitude of 8.1 and the Tangshan earthquake was 1.28 times as intense.

What was the magnitude of the Tangshan earthquake?

For the Mexico earthquake we have:

\[ 8.1 = \log \left( \frac{I_M}{S} \right) \]

For the Tangshan earthquake we have:

\[ M_T = \log \left( \frac{1.28 I_M}{S} \right) \]

Then:

\[ M_T - 8.1 = \log \left( \frac{1.28 I_M}{S} \right) - \log \left( \frac{I_M}{S} \right) = \log \left( \frac{1.28 I_M}{S} \times \frac{I_M}{S} \right) = \log (1.28) = 0.11 \text{ (2 dp)} \]

Then \( M_T \), the magnitude of the Tangshan earthquake is:

\[ 8.1 + 0.11 = 8.21 \]

Activity 4.11

1. The acidity or alkalinity of a solution is measured by its pH. This is the negative of the logarithm to the base ten of the hydrogen ion concentration in moles per litre. The letters pH stand for the potential of hydrogen. Thus \( \text{pH} = -\log(\text{hydrogen ion concentration}) \).

A pH below 7 indicates that a solution is acidic and a pH above 7 indicates that a solution is alkaline. Values are usually between 0 and 14.

a) Find the pH of blood which has a hydrogen ion concentration of 0.000 000 043 moles/litre.

b) If a solution has a pH of 5.25 what is the hydrogen ion concentration?
2. Loudness is measured by comparing the intensity of the sound with the intensity of a sound that is just detected by the human ear.

With $L$ the loudness of the sound in decibels (dB)

$I$ the intensity of the sound

$I_0$ the intensity of the sound just audible to the human ear.

Then $L = 10 \log \left( \frac{I}{I_0} \right)$

How many more times intense is a 90 dB noise level than a 20 dB noise level?

3. The modern logarithmic scale of stellar intensities has a base of $\sqrt[5]{100}$.

If the intensity of star A is given by $I_a$ and the intensity of star B is given by $I_b$, then the formula for the difference between magnitudes is given by:

$b - a = \log_{\sqrt[5]{100}} \left( \frac{I_a}{I_b} \right)$

a) Show that if star A is 100 times as bright as star B, star B is 5 magnitudes greater than star A.

b) What must $\frac{I_a}{I_b}$ be so that star B is 2 magnitudes greater than star A?
Assessment 4

1. Use your calculator to obtain accurately to four decimal places the value of $x$ if:
   
   a) $x = \log 1.287$
   
   b) $10 - 3x = \ln 50$
   
   c) $100^x = 50$
   
   d) $3.98e^x = 24.98$

2. If $\log A = bt + \log P$, express $A$ in terms of the other variables.

3. Using the same scale and axes, sketch the family of curves $y = \log_a x$ for $a = 0.5, 2, 3, e$ and 10.

4. Using the change of base formula, $\left( \frac{\log_a b}{\log_c a} \right)$, evaluate $\log_4 87$ correct to three decimal places.

5. Solve the following exponential equation giving your answer correct to two decimal places: $1.67^x + 1 = 1.98^x$.

6. Solve $\log x + \log (x - 3) = 1$.

7. The size of a colony of bacteria doubles every day.
   
   a) Using the model $A = A_0 e^{kt}$, determine $k$, the percentage increase per hour.
   
   b) If after 10 hours, the size of the colony is 2.5 mm$^3$, calculate the size after 36 hours. (Hint: Calculate the initial size first.)

8. The total number of litres of juice $J$ squeezed from fruit in a vat in $t$ minutes is given by the equation: $J = 6 \ln(28t + 1)$.
   
   a) Find, correct to the nearest mL, the volume of juice squeezed in 2 minutes.
   
   b) Calculate, correct to the nearest second, the number of minutes needed to yield 2 litres.

9. In an experiment to find the luminosity $L$ of a lamp for varying voltage $V$, the following measurements were made.

<table>
<thead>
<tr>
<th>$V$</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
<th>140</th>
<th>160</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>10</td>
<td>31.6</td>
<td>88</td>
<td>184</td>
<td>322</td>
<td>580</td>
</tr>
</tbody>
</table>

   Show that the law is of the type $L = aV^n$, where $a$ and $n$ are constants, and find the value of $a$ and $n$.

10. Show that the following values of $x$ and $y$ are connected by a law of the form $y = ae^{kx}$, and hence find the constants $a$ and $k$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>30.3</td>
<td>47.5</td>
<td>74.5</td>
<td>116.7</td>
<td>182.8</td>
<td>287.1</td>
<td>449.8</td>
</tr>
</tbody>
</table>
Answers to activities

Activity 4.1

1. a) \( \log_3 81 = x \)  
   b) \( \log_y y = 2 \)  
   c) \( \log_{1.44} 1.2 = 0.5 \)  
   d) \( \log_2 0.125 = -3 \)  
   e) \( \log_2 4 = 2 \)  
   f) \( \log_y x = 2 \)  
   g) \( \log_{10} 10 = 1 \)  
   h) \( \log_1 1 = 5 \)  
   i) \( \log_{16} 4 = 0.5 \)  
   j) \( \log_8 4 = 2/3 \)  
   k) \( \log_x 1 = 0 \)  
   l) \( \log_{10} 0.001 = -3 \)

2. a) \( 3^2 = 9 \)  
   b) \( t^0 = 1 \)  
   c) \( 4^{2.5} = 32 \)  
   d) \( x^y = N \)  
   e) \( 2^4 = 16 \)  
   f) \( 4^{0.5} = 2 \)  
   g) \( 12^1 = 12 \)  
   h) \( k^0 = 1 \)  
   i) \( 27^{4/3} = 81 \)

3. a) 6  
   b) 3  
   c) 2  
   d) 1  
   e) 2  
   f) 4  
   g) –2  
   h) 3  
   i) –7  
   j) 0.5  
   k) 0  
   l) –2  
   m) 1.5  
   n) 1.5  
   o) 0.25

4. a) 16  
   b) 27  
   c) 0.25  
   d) 0.01  
   e) 7  
   f) 100  
   g) 1  
   h) 2  
   i) 3  
   j) 2  
   k) a  
   l) Undefined

Activity 4.2

1. a) –1  
   b) 2  
   c) 3  
   d) \( p + q \)  
   e) 7  
   f) –1

2. a) 1  
   b) 1  
   c) 2  
   d) 7  
   e) 2  
   f) 0
Use quadratic, exponential, logarithmic and trigonometric functions and matrices

3. a) 0.7781  
   b) 0.9542  
   c) 1.0791  
   d) 0.2386  
   e) 1.5562  
   f) −0.2219

4. a) 2a  
   b) 10a  
   c) −a  
   d) −3a  
   e) 0.5a  
   f) $\frac{5}{3}a$

5. a) 15  
   b) 4  
   c) 4  
   d) 3

Activity 4.3

1. a) 0.3010  
   b) 1.6094  
   c) 1.3927  
   d) 0.4343  
   e) 2.3026  
   f) 0.8087  
   g) 1.7699  
   h) 1.2301  
   i) 1.2301  
   j) −0.2985  
   k) 116.2767  
   l) 5.5999

2. a) 117.4898  
   b) 0.3360  
   c) 0.0195  
   d) 34.7171  
   e) 1.3802  
   f) 5227.3530  
   g) 0.3679  
   h) 0.0794  
   i) 0.6939

3. a) 1479.1084  
   b) 19.6878  
   c) 2112.1633  
   d) 2.1981  
   e) 0.1121  
   f) 17.7235  
   g) 1.4976  
   h) 1.1580  
   i) 107.5614

Activity 4.4

1. a) 0.6309  
   b) 0.8271  
   c) 1.7898  
   d) 0.6213  
   e) −1.4877  
   f) 1.0726  
   g) 2.2159  
   h) 0.8397  
   i) 1.9470

2. a) $\frac{\log_3 4.5}{\log_3 e}$  
   b) $\frac{\ln 4.789}{\ln 10}$  
   c) $\frac{\log_5 5}{\log_5 2} = \frac{1}{\log_5 2}$

3. $\frac{3b}{a + 2b}$
Activity 4.5

1. 3.9069 2. 4.0857
3. 0.7878 4. 3.7377
5. 1.9795 6. 1.4321
7. 0.0915 8. 1.4849
9. 1.5850 10. 0
11. 3.5046 12. 4.5840
13. –1.0431 14. 4.8100
15. 0.7180 16. 5.7408
17. \(x = 0\) 18. –1.8298
19. –3.6668 20. –4.6747

Activity 4.6

1. \(x = \frac{\log a}{\log b}\) 2. \(x = \log(w - y)\)
3. \(x = 5 - 10^x\) 4. \(x = \ln\left(\frac{z}{k}\right)\)
5. \(x = \frac{k}{10^2}\) 6. \(x = y\)
7. \(d = \frac{2D}{10^{27/276}}\)
Activity 4.7

1.

2.

3.

4.
Section 4 Logarithms

5. Intersection at (0.1111, 0.0458)

6. No intersection
Use quadratic, exponential, logarithmic and trigonometric functions and matrices

7. They are reflections (mirror images) of each other in the line \( y = x \).

8. Intersections at (0.0, -1.0) and (1.4, 0.8)
### Activity 4.8

1. a) 7.61 (2dp)  
   b) 4.52 (2dp)  
2. a) 412.4 (1dp)  
   b) 58.4 (1dp)  
3. 0.045  
4. 6 900 bacteria  
5. 230 bacteria  
6. a) 2%  
   b) 14 800  
7. 12 265 people  
8. a) 62 g  
   b) 78 years  
9. 0.5949 mg  
10. 72.61 °C  
11. 36.79%  
12. 2.23 years  
13. 251 984 bacteria  
14. 47.55 years  
15. 47.99 mg  
16. 35.54%  
17. a) 716 litres  
   b) 20 hours and 38 minutes  
18. 34.7 km/h  
19. No (only 78% healed)  
20. $10 408.11

### Activity 4.9

1. \( a = 4.3 \times 10^{-7}, \ n = 4.1 \)  
2. \( p = 480 \nu^{-1.07} \)  
3. \( a = 5.70, \ n = 0.36 \)  
4. \( \nu^{-1.2} = 240 \)

### Activity 4.10

1. \( A = 10.01 \) (2dp), \( \mu = 0.14 \)  
2. \( a = 10.13 \) (2dp), \( k = -0.71 \)  
3. \( a = 47.528 \) (3dp), \( k = -0.086 \)

### Activity 4.11

1. a) 7.37  
   b) 0.000 00562  
2. 10 000 000 times
3. a) Proof: \( b - a = 5 \)

\[
\therefore \quad 5 = \log_{100} \left( \frac{l_a}{l_b} \right)
\]

\[
\frac{l_a}{l_b} = (100)^5
\]

\[
\therefore \quad = 100
\]

\[
\therefore \quad \text{Star A is 100 times brighter than Star B.}
\]
Section 5 – Trigonometry part 1

5.1 Oblique triangles

An oblique triangle is one without a right angle. It could be an **acute** triangle, with all angles less than 90°, or it could be an **obtuse** triangle in which one angle is more than 90°.

If we want to solve an oblique triangle we cannot use Pythagoras' Theorem or the trigonometric ratios because these methods only apply to right triangles.

However, there are two rules that can be used. They are called the **Sine Rule** and **Cosine Rule**. What rule should be used depends on the information that is provided.

Obviously every triangle has three sides and three angles ie six parts in total. Generally, knowing three of these parts will enable us to solve a triangle, including oblique ones. However, there are exceptions so it is best to consider each combination in turn.

1. Three sides

A triangle is uniquely determined when given three sides. This can easily be demonstrated with a pair of compasses. In geometry, this case is known as the congruency condition **SSS** for Side, Side, Side. Thus when given three side lengths of a triangle we should be able to find the three angle sizes.

2. Two sides and one angle

This case is not as straightforward because it depends on how the angle is positioned in relation to the sides. In fact, there are two possibilities.

a) Two sides and an included angle

Again, using a pair of compasses, it can be shown that a triangle is uniquely determined. The congruency condition is referred to as **SAS** for Side, Angle, Side.

b) Two sides and an angle not included

The SSA case is generally called the **ambiguous** case. The reason is that when given two sides and one angle not included, it is sometimes possible to draw two different triangles, one acute and one obtuse.
Consider this example.

**Example 1** Draw a triangle with $A = 50^\circ$, $a = 6 \text{ cm}$, $b = 8 \text{ cm}$.

Observe the diagram below.

The given information does not define a unique triangle.

In fact, with the given information it is possible to draw two triangles, the obtuse triangle $AB_1C$ and the acute triangle $AB_2C$.

This verifies that when given any two sides and one angle that is not included, we cannot always uniquely solve the triangle. In fact this case is **not** a congruency condition.

Looking at the diagram below, notice that because $CB_1 = CB_2$, angle $B_2$ must be the same size as the external angle $B_1$ of the obtuse triangle $AB_1C$.

Therefore, the obtuse angle $B_1$ in triangle $AB_1C$ is the supplement of the acute angle $B_2$ in triangle $B_1CB_2$, the angles add to $180^\circ$.

This knowledge will help later in solving the two different triangles.

Sometimes on finding $B_2$ and consequently the obtuse angle $B_1$, the size of this obtuse angle $B_1$ added to the size of angle $A$ will produce a total that exceeds $180^\circ$. In such a case, only one possible triangle is defined. In fact, using the configuration as in the diagram, it can be formally shown that two solutions are possible if:

- $A$ is acute, and
- $b \sin A < a < b$. 
Example 2  Verify that if $A = 60^\circ$, $a = 5$ and $b = 6$, only one solution is possible.

Here $b \sin A = 6 \sin 60^\circ = 5.1961…$

This is not less than 5. Hence only one triangle can be drawn (and solved!) using the provided information.

3. One side and two angles
Knowing two angles automatically means that the third angle is known since the angles must add to 180°. When in addition a side length is given, a scale factor is introduced that uniquely determines the size and shape of the triangle. In fact this case is known as the congruency condition **AAS** for Angle, Angle, Side.

4. Three angles
It should be obvious that when given three angles, a triangle is not uniquely determined because different sized (similar!) triangles could be drawn, as is shown.

For this reason, **AAA** could not be used to prove that two triangles are congruent, ie have the same shape and size. Indeed, AAA is not a congruency condition.

Activity 5.1
In each case, three features about a triangle ABC are given. Determine whether the triangle can be solved and if so, whether the solution is unique.

Remember: small letters indicate sides and capital letters indicate angles.

1. $a = 7 \text{ cm}$  $b = 5 \text{ cm}$  $c = 9 \text{ cm}$
2. $A = 34^\circ$  $a = 7 \text{ cm}$  $b = 8 \text{ cm}$
3. $A = 34^\circ$  $B = 78^\circ$  $C = 68^\circ$
4. $A = 34^\circ$  $B = 46^\circ$  $c = 7.4 \text{ mm}$
5. $A = 128^\circ$  $a = 6.3 \text{ m}$  $c = 5.7 \text{ m}$
6. $A = 123^\circ$  $b = 8 \text{ cm}$  $c = 6 \text{ cm}$
5.2 The sine rule

We use the sine rule when given the AAS case and the SSA case.

Consider the following diagram:

Here the perpendicular is dropped from A onto CB.

Now $\sin C = \frac{AP}{b}$ ie $AP = b \sin C$

Also $\sin B = \frac{AP}{c}$ ie $AP = c \sin B$

Thus $b \sin C = c \sin B$ (both equal $AP$)

Rearranging we have: $\frac{b}{\sin B} = \frac{c}{\sin C}$

Now, if we drop a perpendicular from C to AB, as in the diagram below, we have:

$b \sin A = a \sin b$ (both equal to $CQ$)

$\frac{a}{\sin A} = \frac{b}{\sin B}$

Combining the parts, we have the **sine rule**:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
**Example 1** Find, correct to two decimal places, $b$ in triangle ABC if $a = 43.7$, $B = 73^\circ 18'$ and $C = 44^\circ$.

This is an AAS case, thus there is a unique solution.

\[
A = 180^\circ - (B + C) = 62^\circ 42'
\]

\[
b = \frac{a \sin B}{\sin A} = \frac{43.7 \sin 73^\circ 18'}{\sin 62^\circ 42'} \approx 47.10
\]

**Example 2** Correct to the nearest unit and nearest minute, solve triangle ABC, given $c = 314$, $b = 240$ and $C = 55^\circ 10'$.

This is the SSA case so there could be two solutions.

\[
\sin B = \frac{b \sin C}{c} = \frac{240 \sin 55^\circ 10'}{314} = 0.6274...
\]

Then $B = \sin^{-1} 0.6274... = 38^\circ 51'$

Therefore $A = 80^\circ - (B + C) = 85^\circ 59'$
For a:
\[ a = \frac{b \sin A}{\sin B} \]
\[ = \frac{240 \sin 85^\circ 59'}{\sin 38^\circ 51'} \]
\[ = 382 \]

If we check for a possible second solution, we note that \( B \) may be \( 180^\circ - 38^\circ 51' = 141^\circ 10' \) which is impossible for the given triangle because \( 55^\circ 10' + 141^\circ 10' > 180^\circ \).

**Example 3**  Solve the triangle ABC, given \( a = 63 \), \( b = 103.6 \) and \( A = 33^\circ 40' \).

For B:
\[ \sin B = \frac{b \sin A}{a} \]
\[ = \frac{103.6 \sin 33^\circ 40'}{63} \]
\[ = 0.9116.... \]
\[ B = 65^\circ 44' \]

\[ C = 180^\circ - (A + B) \]
\[ = 80^\circ 36' \]

\[ c = \frac{a \sin C}{\sin A} \]
\[ = \frac{63 \sin 80^\circ 36'}{\sin 33^\circ 40'} \]
\[ = 112.1 \]

When \( \sin B = 0.9116 \), \( B \) can also equal \( 180^\circ - 65^\circ 44' = 114^\circ 16' \).
This situation is illustrated in the diagram below.

We have, therefore, another set of solutions.

\[ B = 114^\circ 16' \]
\[ C = 180^\circ - (114^\circ 16' + 33^\circ 40') \]
\[ = 32^\circ 4' \]
\[ c = \frac{63 \sin 32^\circ 4'}{\sin 33^\circ 40'} \]
\[ = 60.3 \]

**Activity 5.2**

Solve the following triangles ABC given that:

1. \( c = 25 \) \( A = 35^\circ \) \( B = 68^\circ \)
2. \( b = 321 \) \( A = 75.3^\circ \) \( B = 38.5^\circ \)
3. \( b = 215 \) \( c = 150 \) \( B = 42.7^\circ \)
4. \( C = 30^\circ \) \( b = 12 \) \( c = 10 \)
5.3 The cosine rule

We use the cosine rule when given the SSS case and the SAS case.

The rule is, referring to the figure below:

![Triangle Diagram]

\[ a^2 = b^2 + c^2 - 2bc \cos A \]
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ c^2 = a^2 + b^2 - 2ab \cos C \]

Rearranging these:

\[ \cos A = \frac{b^2 + c^2 - a^2}{2bc} \]
\[ \cos B = \frac{a^2 + c^2 - b^2}{2ac} \]
\[ \cos C = \frac{a^2 + b^2 - c^2}{2ab} \]

Try to memorise \( a^2 = b^2 + c^2 - 2bc \cos A \) and practise constructing the others above from it. Note that this rule applies even when angle A, B or C is greater than 90°, in which case, the cosine is negative, and the term \(-2bc \cos A\) becomes positive.

The following derivation would help you appreciate the similarity between the cosine rule and Pythagoras’ Theorem. You may skip it if you want to, as we are mainly concerned with the application of the rule.

\[ a^2 = h^2 + d^2 \quad \text{(} h^2 = b^2 - e^2 \text{)} \]
\[ = b^2 - e^2 + d^2 \]
\[ = b^2 - e^2 + (c - e)^2 \quad \text{(} c \text{ being the side opposite to } C \text{)} \]
\[ = b^2 - e^2 + c^2 - 2ce + e^2 \]
\[ = b^2 + c^2 - 2ce \]

But \( e = b \cos A \)

Therefore, \( a^2 = b^2 + c^2 - 2bc \cos A \)
Example 1  Find, correct to the nearest minute, the size of angle $A$ in triangle $ABC$ if $a = 47$, $b = 30$ and $c = 25$.

\[\cos A = \frac{b^2 + c^2 - a^2}{2bc}\]
\[= \frac{(30)^2 + (25)^2 - (47)^2}{2(30)(25)}\]
\[= \frac{900 + 625 - 2209}{(30)(50)}\]
\[= \frac{-684}{1500}\]
\[= (-0.4560)\]
So $A = 117.13^\circ$

Example 2  In triangle $ABC$, $B = 60^\circ$, $a = 4$ cm, $c = 7$ cm. Solve the triangle.

As we are given two sides and the included angle we can therefore apply the cosine rule to find the third side.

\[b^2 = a^2 + c^2 - 2ac \cos B\]
\[= 4^2 + 7^2 - 2 \times 4 \times 7 \times \cos 60^\circ\]
\[= 16 + 49 - 56 \times 0.5\]
\[= 37\]
Therefore $b = \sqrt{37}\]
\[= 6.0828\text{ cm}\]

To find angle $A$, the sine or cosine rule may be used, but the former is more useful.

\[
\sin A = \frac{a \sin B}{b}
\]
\[= \frac{4 \times \sin 60^\circ}{6.0828}\]
\[= \frac{4 \times 0.8660}{6.0828}\]
\[= 0.5695\]
therefore \( A = 34.7° \)
and \( C = 180° - A - B \)
\[ = 180° - 34.7° - 60° \]
\[ = 85.3° \]

Hence:
\( a = 4 \text{ cm and } A = 34.7° \)
\( b = 6.08 \text{ cm and } B = 60° \)
\( c = 7 \text{ cm and } C = 85.3° \)

**Activity 5.3**

Solve the following triangles ABC given that:

1. \( a = 120 \quad b = 270 \quad C = 118.7° \)
   (Note for \( 90° < x° < 180° \), \( \cos x° \) is negative. See later in this Section.)
   The cosine rule is used here initially to find \( c \),
   \[ c^2 = a^2 + b^2 - 2ab \cos C \]
   The sine rule should then be used to complete the solution of the triangle.

2. \( a = 525 \quad c = 421 \quad A = 130.8° \)
3. \( b = 10 \quad c = 12 \quad B = 35° \)
4. \( a = 8 \quad b = 10 \quad c = 9 \)
5.4 Practical oblique triangle problems

Note in the following examples that the first step to solving a problem is to describe it with a clear diagram.

Example 1

A man leaves a point walking at 6.0 km/h in a direction due west. Another man leaves the same point at the same time cycling at a constant speed 28.2 km/h in a direction S 37° W. Find how far the two men are apart after 4 hours.

In the diagram below, \( AB (= c) \) and \( AC (= b) \) represent the distance travelled by the walker and the cyclist respectively. Their distance apart is \( BC (= a) \).

Since distance = (speed) (time)
\( c = 6.0 \times 4 = 24 \text{ km} \)
\( b = 28.2 \times 4 = 112.8 \text{ km} \)

The angle between \( AB \) and \( AC \)
\( = 90° – 37° \)
\( = 53° \)

The required distance, \( BC \), can be found by applying the cosine rule (two sides and the included angle given) to triangle \( ABC \).

\[
a^2 = c^2 + b^2 - 2bc \cdot \cos 53°
\]
\[
= 24^2 + 112.8^2 - 2(24)(112.8) \cos 53°
\]
\[
= 576 + 12723.84 - 3258.47
\]
\[
a = \sqrt{10041.4}
\]
\[
\therefore BC = 100.2 \text{ km}
\]

After 4 hours the two men are 100.2 km apart.
Example 2  If in Example 1 the cyclist’s speed (constant) was unknown and the two men were 120 km apart after 4 hours, find the speed of the cyclist.

Again we can use the triangle ABC to describe the movements of the two men.

In triangle ABC, we have two sides and an angle (not the included one) as the given information:
\[ c = AB = 24 \text{ km} \]
\[ a = BC = 120 \text{ km} \]
\[ A = 53^\circ \]

Thus the sine rule should be used.
\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]

We are required to find \( b \), so that the cyclist speed can be calculated.

We use
\[ \frac{b}{\sin B} = \frac{a}{\sin A} \quad \text{or} \quad \frac{b}{\sin B} = \frac{c}{\sin C} \]

But either of the above contains two unknown quantities \( b \) and \( \sin B \). Therefore we need to find \( \sin B \) first as follows:

From
\[ \frac{a}{\sin A} = \frac{c}{\sin C} \]
we have
\[ \frac{120}{\sin 53^\circ} = \frac{24}{\sin C} \]
\[ \sin C = \frac{24 \sin 53^\circ}{120} = 0.1597 \]
\[ C = 9.2^\circ \]

(or 170.8, NOT possible because 170.8 + 53° > 180°)
Thus there is only one possible solution for triangle ABC.

\[ B = 180° - (A + C) \]
\[ = 180° - (53° + 9.2°) \]
\[ = 117.8° \]

Hence:
\[ \frac{b}{\sin 117.8°} = \frac{120}{\sin 53°} \]

Then
\[ b = \frac{120 \sin 117.8°}{\sin 53°} \]
\[ = 132.9. \]

Hence the speed of the cyclist is \[ \frac{132.9}{4} = 33.2 \text{ km/h}. \]

Example 3
Walking towards a tower at 8 km an hour, a man observed at a particular time that the angle of elevation of its top was 10.2°, and 3 minutes afterwards that it was 32°. Find the tower’s height and distance from the second point of observation.

The angle of elevation of the tower top from an observer is the angle between the line of sight and the horizontal, as shown in the diagram.

In the diagram (NOT to scale):

A and B are the first and second points of observation respectively.

CD is the tower, whose height, \( h \) is required.

Before we can use the triangle ABC (or BCD), to find \( h \) and the required distance, BD, we need to know \( b \) (or \( a \)).

We will use triangle ABC to find \( b \), with the following information:

\[ A = 10.2° \]
\[ c = AB = \text{speed} \times \text{time} \]
\[ = 8 \times \frac{3}{60} \left( Time = \frac{3}{60} \text{ hour} \right) \]
\[ = 0.4 \text{ km} \]

\[ \text{angle } ACB = 180° - 32° = 148° \]
With two angles and a side given, the sine rule should be used.

Since \( c = 0.4 \), we want: angle \( ACB = 180^\circ - (10.2^\circ + 148^\circ) = 21.8^\circ \)

The sine rule gives:

\[
\frac{c}{\sin 21.8^\circ} = \frac{b}{\sin 148^\circ}
\]

\[
b = \frac{0.4 \sin 148^\circ}{\sin 21.8^\circ}
\]

\[
= 0.5708
\]

Therefore: \( h = b \sin 10.2^\circ \)

\[
= 0.57 \sin 10.2^\circ
\]

\[
= 0.10
\]

\( AD = b \cos 10.2^\circ \)

\[
= 0.56
\]

\( BD = 0.56 - 0.4 \)

\[
= 0.16
\]

The height of the tower is 0.1 km (or 100 m) and it is 0.16 km from the second point of observation.

**Example 4**

A ship is observed from the top of a cliff, 152 m high, in a direction S 28° 19' W at an angle of depression 8° 46'. Six minutes later, the same ship is seen in a direction W 17° 13' N at an angle of depression 9° 52'. Calculate the speed of the ship.
In $\triangle AXB$ \[ \tan 8^\circ 46' = \frac{152}{BX} \rightarrow BX = \frac{152}{\tan 8^\circ 46'} = 985.65 \text{ m} \]

In $\triangle AYB$ \[ \tan 9^\circ 52' = \frac{152}{BY} \rightarrow BY = \frac{152}{\tan 9^\circ 52'} = 873.92 \text{ m} \]

Using the cosine rule in $\triangle XYB$:

Angle $B = 17^\circ 13' + (90 - 28^\circ 19')$

$= 17^\circ 13' + 61^\circ 41'$

$= 78^\circ 54'$

$XY^2 = 985.65^2 + 873.92^2 - 2(985.65)(873.92) \cos 78^\circ 54'$

$XY = 1184.72 \text{ m}$

So the ship has sailed 1.185 km in 6 min or 11.85 km in 60 min.

So the average speed is 11.85 km/h.

Example 5

From a point $P$ at ground level, a mine shaft is constructed to a depth of 0.264 km. Tunnels are constructed from $Q$, 3.450 km due west of $P$ and from $R$, 2.875 km and $S$ 40° E of $P$, to meet at the base $S$ of the shaft. Assuming $P$, $Q$ and $R$ are on horizontal ground and the two tunnels descend uniformly, calculate their angles of descent and the angle between the directions of the tunnels.

In $\triangle QRS$ \[ \tan Q = \frac{0.264}{3.450} = 0.0765 \]

$\angle Q = 4.38^\circ$

In $\triangle PRS$ \[ \tan R = \frac{0.264}{2.875} = 0.0918 \]

$\angle R = 5.25^\circ$

In $\triangle PQR$ \[ QR^2 = 3.450^2 + 2.875^2 - 2(3.450)(2.875) \cos 130^\circ \]

$QR^2 = 32.92$

$QR = 5.738 \text{ km}$
\[ QS^2 = 3.450^2 + 0.264^2 \]
\[ QS^2 = 11.972 \]
\[ QS = 3.460 \text{ km} \]
and
\[ RS^2 = 2.875^2 + 0.264^2 \]
\[ RS^2 = 8.335 \]
\[ RS = 2.887 \text{ km} \]

In \( \triangle QRS \)

\[
\cos \angle S = \frac{3.46^2 + 2.887^2 - 5.738^2}{2(3.46)(2.887)}
\]
\[
= \frac{12.618}{19.987}
\]
\[
= -0.6316
\]

Hence \( \angle S = 129^\circ 17' \)

**Activity 5.4**

1. An architect designing a building wants the roof overhang \( AB \) on the north side such that in mid-summer when the noon altitude of the sun is 80°, the 10 m wall \( AC \) is just in shadow.

Find the length \( AB \) to the nearest 0.01 m given angle \( CAB \) has size 70°.

![Diagram of a roof overhang](image)

2. A and B are two points on opposite banks of a river. From A, a line \( AC = 275 \text{ m} \) is laid off and the angles \( CAB = 125^\circ 40' \) and \( ACB = 48^\circ 50' \) are measured. Find the length of \( AB \).

3. From A, a pilot flies 125 km in the direction N 38° 20' W and then turns back. Through an error he then flies 125 km in the direction S 51° 40' E. How far and approximately in what direction must he now fly to reach his intended destination A?

4. A and B are two points 650 m apart on one bank of a straight river. C is a point on the other bank, and angles \( CAB, CBA \) are observed to be 46° 23' and 67° 38' respectively. Find the width of the river.
5. AB is the base line of a survey, C and D two points on the same side of AB which are visible from both A and B. If AB is 4 km long and DAB = 22° 15', DBA = 35° 30'. CAB = 65°, CBA = 78° 30', find by calculation the distance CD and angle CDA.

6. A man observes two towers each 10 metres high. The bearing of the first tower is N 15° E and the angle of elevation of the top of the tower is 22° 15'. The bearing of the second tower is N 70° E and the angle of elevation is 18° 42'. Find the distance between the two towers.

7. A coastguard situated at the top of a cliff 200 m high observes a ship in a direction S 32° W at an angle of depression of 9° 14'. Five minutes later the same ship is seen in a direction W 25° N at an angle of depression of 10° 51'. Calculate the speed of the ship in km/h⁻¹.

5.5 Radian measure

There is another way of measuring angles which is used extensively in higher mathematics. This method has as its unit the radian which is the standard angular measure in the International System of Units (S.I.).

By definition, 1 radian is the angle between two radii of a circle which cut off (subtends) on the circumference an arc equal in length to one radius.

Although the circles in the above diagram are of different sizes, the angles subtended at centres are all 1 radian (1 rad or 1R) in size.

Radians must be used in certain formulas in mathematics and physics to avoid errors or the need for additional scale factors.

Because of this widespread use, the rad abbreviation or the R sign is generally omitted.

To avoid confusion, it has been agreed that the degree sign ° must always be written. Thus when we may have to calculate sin 2, for example, it means that we need to calculate the sine of 2 radians, not 2 degrees.

From the diagram you can see that 1 radian must be approximately 60°. In fact, it is approximately 57.3°. The exact conversion is easily derived if you consider that an angle of 360° must have size $2\pi$ rad since the length of the circumference equals $2\pi$ radius units.
Section 5

Trigonometry part 1

Thus $2\pi \text{ rad} = 360^\circ$ or $\pi \text{ rad} = 180^\circ$

or $1 \text{ rad} = \frac{180^\circ}{\pi}$ after dividing both sides by $\pi$.

Considering that $\pi \approx 3.1415926\ldots$, we obtain $1 \text{ rad} = 57.295779\ldots^\circ$

$= 57^\circ 17' 45''$ to the nearest second.

The relationship $\pi \text{ rad} = 180^\circ$ holds the key to converting between the measures because with it the conversion factors can easily be derived. They are summarised in the following table.

<table>
<thead>
<tr>
<th>Converting from</th>
<th>Radians to degrees</th>
<th>Degrees to radians</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\times \frac{180^\circ}{\pi}$</td>
<td>$\times \frac{\pi}{180^\circ}$</td>
<td></td>
</tr>
</tbody>
</table>

**Example 1** Change 4.35 radians to degrees, minutes and seconds.

$4.35^R = 4.35^R \times \frac{180^\circ}{\pi} = 249.23664\ldots^\circ$

$= 249^\circ 14' 12''$ (nearest second)

**Example 2** Convert $36^\circ 17' 35''$ to radians correct to 4 decimal places.

$36^\circ 17' 35'' = 36.2930555\ldots^\circ$

$= 36.2930555\ldots^\circ \times \frac{\pi}{180^\circ} = 0.6334^R$ (correct to 4 dp)

**Activity 5.5**

1. Convert to radian measure, correct to 4 decimal places.
   a) $34^\circ$     b) $145.81^\circ$     c) $12^\circ 14'$
   d) $23^\circ 15' 25''$     e) $145^\circ 12' 54''$     f) $56^\circ 15' 34.7''$

2. Convert to degrees, correct to the nearest second.
   a) $2^R$     b) $4.1^R$     c) $2.8765^R$
   d) $\frac{1}{2}\pi^R$     e) $\frac{\pi}{6}$     f) $\frac{2\pi^R}{3}$

[Attribution: BY-NC-SA]
3. Write down the exact radian equivalent (ie expressed in terms of \( \pi \)) such as \( 30^\circ = \frac{\pi}{6} \) for the following:
   a) 45°  b) 60°  c) 90°  
   d) 120°  e) 225°  f) 330°  

5.6 Applications of radian measure

As mentioned previously, radians must be used in certain formulae in mathematics and physics to avoid errors or the need for additional scale factors.

We shall look at three formulae in which radians are used.

1. Length of an arc

An arc of a circle will have a length that is in proportion to the angle subtended.

In a full circle the arc length is the circumference which has a length of \( 2\pi r \) and the angle subtended is \( 2\pi \) rad. If we let \( s \) be any arc length which subtends an angle \( \theta \), then

\[
\frac{s}{\theta} = \frac{2\pi r}{2\pi}
\]

Cancelling and re-arranging, gives us the formula \( s = r \theta \).

Example 1  Calculate the perimeter of the sector drawn below using 3 significant figures.

\[
33^\circ = 33^\circ \times \frac{\pi r}{180^\circ} = 0.57595\ldots R
\]

Then \( s = 6 \times 0.57595\ldots = 3.46 \text{ mm} \) (3 SF)  
Thus the perimeter is \( 6 + 6 + 3.46 = 15.46 \text{ mm} \).
2. Area of a sector

A sector in a circle will have an area in proportion to the angle subtended.

For a full circle the area is \( \pi r^2 \) and the angle subtended is \( 2\pi \) rad. If we let \( A \) be the area of a sector with subtended angle \( \theta \), then

\[
\frac{A}{\theta} = \frac{\pi r^2}{2\pi}
\]

Cancelling and re-arranging, gives us the formula

\[
A = \frac{1}{2} r^2 \theta
\]

**Example 2** Find, correct to 4 decimal places, the area of the sector with radius 0.8 cm and subtended angle 80°.

\[80° = 80 \times \frac{\pi}{180} = \frac{4\pi}{9}\]

Then area of sector

\[
= \frac{1}{2} \times \frac{4\pi}{9} (0.8)^2 \text{ cm}^2
\]

\[= 0.4468 \text{ cm}^2\]

3. Angular displacement and velocity

Consider this problem to illustrate the concept of **angular displacement**:

**Example 3** A pulley of diameter 0.5 m is rotated by a belt. If the belt revolves once while the pulley rotates 15 times, how long is the belt?

If the diameter of the pulley is 0.5 m then the radius is 0.25 m.

Also 15 rotations \( = 30 \pi \)R

Let \( b \) be the length of the belt, then \( b \) is the circular distance travelled by a point on the circumference of the pulley.

Using \( s = r \theta \), we obtain \( b = 0.25 \times 30 \pi R = 23.6 \) (correct to 1 dp).

To determine angular velocity, recall that when an object moves in a straight line with constant speed, \( v \), then \( s = vt \) where \( s \) is the distance travelled, and \( t \) is the time taken.

Thus \( v = \frac{s}{t} \)

If we think of \( s \) as a distance travelled around a circle \( (s = r \theta) \)

then \( v = \frac{r \theta}{t} \)

So \( v = r \times \frac{\theta}{t} = r \omega \)

where \( \omega = \frac{\theta}{t} \) is the angular velocity at which a point is turning.
Example 4  A pulley wheel turns at 50 rad/s. Its diameter is 12 cm. Find the velocity of a point on the rim of the pulley.

Here \( r = 6 \text{ cm} \) and \( \omega = 50 \text{ rad/s} \)

\[
V = r\omega = (6 \text{ cm})(50 \text{ rad/s}) = 300 \text{ cm/s}
\]

Activity 5.6
1. A pendulum clock sweeps through an angle of 15°. If the length of the pendulum arm is 1.02 m and the pendulum itself is 20 cm, calculate to the nearest cm the length of the arc swept by the midpoint of the pendulum.
2. An arc of length 6 cm subtends an angle of 56° at the centre of a circle. Find the length of the radius using the formula \( s = r\theta \).
3. A drill bit rotates at 2000 rpm. If it is 10 mm in diameter, what is its angular velocity? What is the linear velocity of a point on its rim?
4. A truck with tyres of radius 1.2 m travels at 40 km/h. Find the angular velocity of the tyres in rad/h.
5. A circular saw rotates at 20 rpm. If the blade is 16 cm in diameter, calculate the speed of one of the teeth in cm/sec.
6. Find the area of the sector of a circle shaded in the diagram.
5.7 The unit circle

You should be familiar with the coordinate plane which is used in algebra for graphing sets of points. You will remember that the plane was divided into four quarter planes, called quadrants, by the axes.

In order to define the trigonometric ratios for non acute angles, a circle (the unit circle) is drawn on this plane with centre at the origin and a radius of one unit.

In addition, an angle is drawn with one ray (the initial ray) on the positive x axis. As is shown in the diagram, the other ray (the terminal ray) will intersect the unit circle in a point \( P(x, y) \). The angle is said to be in standard position.

Using the unit circle, an angle of any size, even a negative angle, can be defined. Formally these types of angles are called rotational angles.

When the angle is in standard position, the initial ray coincides with the positive x axis. If we rotate the terminal ray in an anti-clockwise direction we produce positive angles while if we rotate in a clockwise direction we produce negative angles.
Example  Illustrate a rotational angle of \((-294^\circ)\).

Note that the angle is equivalent to a positive rotational angle of \(66^\circ\).

Activity 5.7

Draw angles in standard position with the following rotational sizes. Mark your angle to show the direction of rotation.

1. \(145^\circ\)
2. \((-90^\circ)\)
3. \((-235^\circ)\)
4. \(190^\circ\)
5. \(\pi\)
6. \(\frac{\pi}{2}\)
7. \(-\frac{3\pi}{2}\)
8. 2.1
5.8 Definition of sine and cosine

We are now in a position to define the two fundamental trigonometric functions sine and cosine of an angle. Referring to the unit circle:

\[ x = \cos \theta \]
\[ y = \sin \theta \]

The y coordinate of point P is called the sine of the angle, ie \( y = \sin \theta \).

The x coordinate of point P is called the cosine of the angle, ie \( x = \cos \theta \).

Example 1  Refer to the unit circle diagram below, in which an angle of 37° is drawn.

Estimate the coordinates of the point P\((x, y)\) and hence state the value of \( \sin 37^\circ \) and \( \cos 37^\circ \).

From the figure the coordinates of P are approximately (0.8, 0.6).
Hence \( \sin 37^\circ = 0.6 \) and \( \cos 37^\circ = 0.8 \) (both correct to 1 dp).

Because sine and cosine are defined using a (unit) circle, these functions are generally referred to as circular functions.
Let us look at further examples in which you notice that negative values for sine and cosine can be obtained.

**Example 2** Using a unit circle, write down the approximate values of:

(i) \( \sin 140^\circ \)  
(ii) \( \cos 220^\circ \)  
(iii) \( \sin (-50^\circ) \)

(i) \( \sin 140^\circ = 0.65 \)

(ii) \( \cos 220^\circ = (-0.75) \)

(iii) \( \sin (-50^\circ) = (-0.775) \)
### Activity 5.8

Using a unit circle and your protractor, find the approximate values of sine and cosine of the following angles.

1. 60°
2. 80°
3. 45°
4. 90°
5. 130°
6. 110°
7. 20°
8. 180°
9. 225°
10. 270°
11. (–60°)
12. 360°

### 5.9 Other trigonometric functions

From your previous studies in trigonometry you should know that \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)

Hence in terms of the coordinates of \( P \) we can define

\[ \tan \theta = \frac{y}{x} \]

Also in terms of the coordinates of \( P \) we can define

\[ \sec \theta = \frac{1}{x}, \quad \cosec \theta = \frac{1}{y} \quad \text{and} \quad \cot \theta = \frac{x}{y} \]

**Example 1**

Given that \( \sin 37° \approx 0.6 \) and \( \cos 37° \approx 0.8 \), obtain \( \tan 37°, \cot 37°, \cosec 37° \) and \( \sec 37° \) and correct to 2 decimal places.

\[
\begin{align*}
\tan 37° &= \frac{0.6}{0.8} = 0.75 \\
\cot 37° &= \frac{0.8}{0.6} = 1.33 \\
\sec 37° &= \frac{1}{0.8} = 1.25 \\
\cosec 37° &= \frac{1}{0.6} = 1.67
\end{align*}
\]

Normally we would use a calculator to find trigonometric values. Before using a calculator we must put it in the correct mode. You may wish to consult your instruction booklet on how to change modes.

We do two examples using degrees first. You should verify the answers using your own calculator.
Example 2  Obtain the following correct to 4 decimal places.

a) \( \sin(-16^\circ) \)  
  \[ \approx -0.2756 \]

b) \( \sin 121^\circ \)  
  \[ \approx 0.8572 \]

c) \( \cos(-100.5^\circ) \)  
  \[ \approx -0.1822 \]

d) \( \tan 256.75^\circ \)  
  \[ \approx 4.2468 \]

e) \( \cos 12^\circ17'23'' \)  
  \[ \approx 0.9771 \]

f) \( \sin(-12^\circ15'45'') \)  
  \[ \approx -0.2124 \]

In order to determine values of sec, cosec and cot use the reciprocal button \( \frac{1}{x} \).

Example 3  Find sec 37°:

First find \( \cos 37^\circ = 0.79863551\ldots \) and then apply the definition

\[
\frac{1}{\cos 37^\circ} = \sec 37^\circ \quad \text{so} \quad \sec 37^\circ = \frac{1}{0.7986355\ldots} = 1.2521
\]

If you wish to find the trigonometric functions of angles measured in radian measure the calculator must be in radian mode.

You should verify the answers given in the following examples.

Example 4  Obtain the following correct to 4 decimal places:

a) \( \sin 2 \)  
  \[ \approx 0.9093 \]

b) \( \sin(-5) \)  
  \[ \approx 0.9589 \]

c) \( \cos 0.6 \)  
  \[ \approx 0.8253 \]

d) \( \tan 1.35 \)  
  \[ \approx 4.4552 \]

e) \( \cosec 0.25 \)  
  \[ \approx 4.0420 \]

f) \( \cot 1.55\pi \)  
  \[ \approx -0.1584 \]

Activity 5.9

1. Complete the following table. Answer correct to 4 decimal places.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>30(^\circ)</th>
<th>(-75(^\circ))</th>
<th>1.2</th>
<th>(-2.6)</th>
<th>140(^\circ)</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin ) ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cos ) ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \tan ) ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cosec ) ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sec ) ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cot ) ( \theta )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Find all other trigonometric ratios given that \( \sin 53^\circ = 0.8 \) and \( \cos 53^\circ = 0.6 \).

3. Obtain the following correct to 4 decimal places.
   a) \( \sin 2.12 \)  
   b) \( \sin (-50.45^\circ) \)  
   c) \( \csc (-0.6) \)  
   d) \( \cot 1.3523 \)  
   e) \( \sec 0.25^\circ \)  
   f) \( \cot 34^\circ 12' 45'' \)

4. Obtain the following correct to 4 decimal places.
   a) \( \sin^2 2.5 \)  
   b) \( \sec (\cos \pi) \)  
   c) \( \csc^2 14^\circ 12' 46'' \)  
   d) \( \frac{1}{\tan 12^\circ} \)  
   e) \( \tan^2 1 \)  
   f) \( \cos \left( \frac{\pi}{4} \right) \)

5.10 Signs of the trigonometric functions

In using the calculator in the previous topics, you noticed that some answers were positive and others were negative. This is not surprising because the trigonometric quantities depend on where the terminal ray intersects with the unit circle. Only in the first quadrant are both the \( x \) (ie the cosine) and the \( y \) coordinate (ie the sine) of the intersection point positive.

In the following table the four different scenarios are illustrated.
Use quadratic, exponential, logarithmic and trigonometric functions and matrices

**Quadrant 3**

\[ \pi < t < \frac{3\pi}{2} \]

- \( \cos t = x \) is negative
- \( \sin t = y \) is negative
- \( \tan t = \frac{y}{x} \) is positive

**Quadrant 4**

\[ \frac{3\pi}{2} < t < 2\pi \]

- \( \cos t = x \) is positive
- \( \sin t = y \) is negative
- \( \tan t = \frac{y}{x} \) is negative

**Note:** Signs for cosec, sec and cot ratios are the same as for sin, cos and tan ratios respectively.

When the signs of the three trigonometric functions in the different quadrants are considered we notice that All are positive in the first quadrant, only the \textbf{S}ine is positive in the second quadrant, only the \textbf{T}angent is positive in the third quadrant and only the \textbf{C}osine is positive in the fourth quadrant. You may wish to remember this as ‘\textbf{A}ll \textbf{S}tudents \textbf{T}ake \textbf{C}lasses’ or by the mnemonic \textbf{CAST}. 

\[
\begin{array}{c|c|c}
\pi & 0,2\pi & 0 \\
\hline
S & A & T \\
\hline
\frac{\pi}{2} & & \\
\hline
\frac{3\pi}{2} & & \\
\end{array}
\]
In some problems, in which the calculator is of limited use, knowledge of the sign is essential to provide the correct answer. Consider this example.

Example 
If \( \sin A = \frac{3}{5}, \; \frac{\pi}{2} \leq A \leq \pi \), find the value of the other circular functions.

Use \( \sin^2 A + \cos^2 A = 1 \)

Then \( \sin^2 A + \cos^2 A = 1 \)
\( \cos^2 A = 1 - \sin^2 A \)
\( = 1 - \left( \frac{3}{5} \right)^2 \)
\( \cos^2 A = \frac{16}{25} \)
So \( \cos A = \pm \sqrt{\frac{16}{25}} = \pm \frac{4}{5} \)

Since \( \frac{\pi}{2} \leq A \leq \pi \), \( \cos A < 0 \)
So \( \cos A = \left( \frac{-4}{5} \right) \)

Then \( \tan A = \frac{\sin A}{\cos A} = \frac{3}{\frac{-4}{5}} = \left( \frac{-3}{4} \right) \)
\( \sec A = \frac{1}{\cos A} = \frac{1}{\left( \frac{-4}{5} \right)} = \left( \frac{-5}{4} \right) \)
\( \cosec A = \frac{1}{\sin A} = \frac{1}{\frac{3}{5}} = \frac{5}{3} \)
\( \cot A = \frac{1}{\tan A} = \frac{1}{\left( \frac{-3}{4} \right)} = \left( \frac{-4}{3} \right) \)

Activity 5.10
Find the value of the other five trigonometric functions if:

1. \( \cos x = \frac{12}{13} \) and \( 0 < x < \frac{\pi}{2} \)  
2. \( \tan x = \frac{3}{5} \) and \( 180^\circ < x < 270^\circ \)
3. \( \sec x = (-3) \) and \( \pi < x < \frac{3\pi}{2} \)
4. \( \cosec x = (-2) \) and \( \frac{3\pi}{2} < x < 2\pi \)
5.11 Solving trigonometric equations

Previously when we encountered equations such as \( \sin x = 0.87 \) we always assumed that \( x \) was an acute angle. Now that we know that \( x \) can be any size, in order to solve equations such as the above, we need to know the restrictions on \( x \) (the domain values).

The unit circle can be used to solve this type of equation. The process involves finding the reference angle. This angle, generally designated as \( \alpha \), is the acute angle a rotational angle \( \theta \) makes with the \( x \) axis.

Next the trigonometric value of the reference angle is obtained and the appropriate sign according to the quadrant in which the terminal ray lies is attached.

As usual we shall illustrate the process by means of some examples. Note, however, that trigonometric equations can also be solved using trigonometric graphs which are covered later in this Section. Because this graphical approach is more universal and easier to use we shall concentrate on that method.

**Example 1** Solve \( \sin \theta = 0.7547 \) to the nearest degree if \( 0^\circ \leq \theta \leq 360^\circ \).

\[
\begin{align*}
\text{Here the reference angle (to the nearest degree) is:} \\
\alpha &= \sin^{-1} 0.7547 = 49^\circ
\end{align*}
\]

From the unit circle diagram it is clear that the reference angle is one solution.

However, there is another solution which is obtained from subtracting the reference angle from \( 180^\circ \). This angle is in quadrant 2 where the sine is also positive.

Hence \( \theta = 49^\circ \) or \( 131^\circ \).
Example 2  Solve \( \cos \theta = -0.7260 \) to two decimal places if \(-2\pi \leq \theta \leq 0\).

Here the reference angle (to two decimal places) is:
\[ \alpha = \cos^{-1} 0.7260 = 0.76 \]

We have to make negative rotations and obtain angles in quadrants 2 and 3 where the cosine is negative.

From the diagram we need the angles
\[ -\pi + 0.76 = -2.38 \]
and \(-\pi - 0.76 = -3.90\).

Hence \( \theta = (-2.38) \) or \((-3.90)\).

Activity 5.11
1. State the reference angle for the following angles.
   a) \(140^\circ\)  b) \(215^\circ\)  
   c) \((-230^\circ)\)  d) \(72^\circ\)  
   e) \(670^\circ\)  f) \(3.7654\)  
   g) \((-2.25)\pi\)  h) \((-22.35)\pi\)

2. Use your calculator to find the acute angle \(\theta\), correct to the nearest second, given:
   a) \(\sin \theta = 0.9468\)  b) \(\sin \theta = 0.8899\)  
   c) \(\tan \theta = 0.5200\)  d) \(\tan \theta = 2.7775\)  
   e) \(\cot \theta = 0.6200\)  f) \(\sec \theta = 3.9382\)
3. Use your calculator to find the acute angle $\theta$, correct to 4 decimal places, given:
   a) $\sin \theta = 0.9468$  
   b) $\sin \theta = 0.0899$
   c) $\tan \theta = 0.3470$  
   d) $\tan \theta = 6.7775$
   e) $\cot \theta = 0.8200$  
   f) $\sec \theta = 1.9382$

4. Solve the following correct to the nearest second.
   a) $\sin \theta = 0.8654$ for $0^\circ \leq \theta \leq 360^\circ$  
   b) $\cos \theta = (-0.8765)$ for $0^\circ \leq \theta \leq 360^\circ$
   c) $2\sin \theta = 1.2342$ for $0^\circ \leq \theta \leq 90^\circ$  
   d) $\sin \theta = 0.9871$ for $-90^\circ \leq \theta \leq 90^\circ$
   e) $\cos \theta = 0.9876$ for $-180^\circ \leq \theta \leq 180^\circ$  
   f) $\sin \theta = (-0.2156)$ for $-360^\circ \leq \theta \leq 360^\circ$

5. Solve the following correct to 3 decimal places.
   a) $\sin \theta = 0.9876$ for $0 \leq \theta \leq \pi$  
   b) $\cos \theta = 0.3210$ for $0 \leq \theta \leq 2\pi$
   c) $\sec \theta = 1.2367$ for $0 \leq \theta \leq \pi$  
   d) $-2\cosec \theta = 2.3215$ for $-\pi \leq \theta \leq \pi$
   e) $\frac{1}{\cosec \theta} = 0.7761$ for $0 \leq \theta \leq \pi$  
   f) $\cos^2 \theta - 0.2\pi = 0$ for $0 \leq \theta \leq 2$
Assessment 5

1. Change 2.456 radians to degrees, minutes and seconds.

2. What in terms of $\pi$ is the radian equivalent of 135$^\circ$?

3. In the diagram calculate:

   \[ \triangle ABC \]

   \[ \angle B = 30^\circ \]
   \[ BC = 0.9 \text{ cm} \]

   a) the size of $\angle B$ in radians
   b) the arc length $AC$
   c) the area of sector $ABC$

4. If a wheel makes three revolutions in covering 18 metres of distance along the ground, what is the radius of the wheel?

5. An armature turns $\frac{3\pi}{2}$ radians. How many degrees has it turned through?
   How many revolutions has it made?

6. A wheel turns at 30 revolutions per minute. How many radians per second is this?

7. A disc revolves through 6 revolutions in 0.5 min. What is the angular velocity?
   If the radius is 5 cm, what is the linear velocity of a hole punched near the rim of the disc?

8. Explain why the SSA case is called the ambiguous case.

9. Solve triangle $ABC$ with $A = 45^\circ$, $b = 6$ mm and $c = 7$ mm.

10. In the triangle $ABC$ with $\angle ABC = 64^\circ$, $AB = 35$ mm and $CA = 48$ mm.
    Calculate the size of $\angle CAB$.

11. In the triangle shown, calculate the length $p$ of side $QR$.

   \[ \triangle PQR \]

   \[ \angle P = 39^\circ 15' \]
   \[ PQ = 27 \text{ cm} \]
   \[ QA = 19 \text{ cm} \]
12. A tunnel is to be dug through a mountain from point A to point C. From a point B both points A and C are visible. If the angle ABC is measured as 57° 15' and if B is 1036 m from A and 1348 m from C calculate the length of the proposed tunnel.

13. A vertical mast stands on horizontal ground. A surveyor standing due south of the mast measures the angle of elevation of the top as 37° 28'. He walks in a direction N 79° 40' W to a point 72 m from the base of the mast which is now on a bearing of N 17° 50' E. Calculate the height of the mast.


15. Draw an accurate graph of the unit circle and estimate the value of cos 45°.

16. Use a calculator to obtain correct to 4 dp:
   a) sin 325°  b) tan 2.5  c) sin² 5.2

17. Calculate correct to 4 dp:
   a) sec 2  b) cosec 12  c) cot 12° 11' 13"

18. Find the values of the other five trigonometric functions if:
    \[ \sin x = \frac{3}{5} \] and x is in the second quadrant.

19. Solve using the unit circle the equation 1 − 2\cos x = 0.25 for −360° ≤ x ≤ 360°. Answer correct to the nearest second.

20. Solve using the unit circle and correct to 2 decimal places:
    \[ \sin x = (-0.9) \] for 0 ≤ x ≤ 2π.
Section 5 Trigonometry part 1

Answers to activities

Activity 5.1
1. Yes, unique  
2. Yes, not unique  
3. No, cannot be solved  
4. Yes, unique  
5. Yes, unique

Activity 5.2
1. \( C = 77^\circ \quad a = 14.72 \quad b = 23.79 \)  
2. \( C = 66.2^\circ \quad c = 471.8 \quad a = 498.8 \)  
3. \( C = 28.2^\circ \quad A = 109.1^\circ \quad a = 299.7 \)  
4. \( B = 36.9^\circ \quad A = 113.1^\circ \quad a = 18.39 \)  
   or \( B = 143.1^\circ \quad A = 6.9^\circ \quad a = 2.39 \)

Activity 5.3
1. \( c = 344.1 \quad A = 17.8^\circ \quad B = 43.5^\circ \)  
2. \( C = 37.4^\circ \quad B = 11.8^\circ \quad b = 142.1 \)  
3. \( C = 43.5^\circ \quad A = 101.5^\circ \quad a = 17.08 \)  
4. \( A = 49.5^\circ \quad B = 71.8^\circ \quad C = 58.7^\circ \)  
   or \( C = 136.5^\circ \quad A = 8.5^\circ \quad a = 2.58 \)

Activity 5.4
1. 1.76 m  
2. 2160 m  
3. 29.02 km south west  
4. 476 m  
5. 4.938 km, 115° 4'  
6. 26.9 m  
7. 18.16 km/h

Activity 5.5
1. a) 0.5934  
   b) 2.5449  
   c) 0.2135  
   d) 0.4059  
   e) 2.5345  
   f) 0.9819

2. a) 114° 35' 30"  
   b) 234° 54' 46"  
   c) 164° 48' 41"  
   d) 90°  
   e) 30°  
   f) 120°

3. a) \( \frac{\pi}{4} \)  
   b) \( \frac{\pi}{3} \)  
   c) \( \frac{\pi}{2} \)  
   d) \( \frac{2\pi}{3} \)  
   e) \( \frac{5\pi}{4} \)  
   f) \( \frac{11\pi}{6} \)

Activity 5.6
1. 29 cm  
2. 6.14 cm  
3. 209.44 rad/s 1047.2 mm/s  
4. 33333 \( \frac{1}{3} \) rad/h  
5. 16.8 cm/s  
6. 1.32 m²
Activity 5.7

1. \[ \begin{align*} &\text{\(145^\circ\)} \\
&\text{\((-90^\circ)\)}
\end{align*} \]

2. \[ \begin{align*} &\text{\((-235^\circ)\)} \\
&\text{\(190^\circ\)}
\end{align*} \]

3. \[ \begin{align*} &\text{\(\pi\)} \\
&\text{\(\frac{\pi}{2}\)}
\end{align*} \]

4. \[ \begin{align*} &\text{\((-\frac{3\pi}{2})\)} \\
&\text{\(2.1\)}
\end{align*} \]
Section 5 Trigonometry part 1

Activity 5.8
1. 0.87, 0.5  
2. 0.98, 0.17  
3. 0.71, 0.71  
4. 1, 0  
5. 0.76, –0.64  
6. 0.94, –0.34  
7. 0.34, 0.94  
8. 0, –1  
9. –0.71, –0.71  
10. –1, 0  
11. –0.87, 0.5  
12. 0, 1

Activity 5.9
1.

<table>
<thead>
<tr>
<th>θ</th>
<th>30°</th>
<th>(–75°)</th>
<th>1.2</th>
<th>(–2.6)</th>
<th>140°</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin θ</td>
<td>0.5</td>
<td>(–0.9659)</td>
<td>0.9320</td>
<td>(–0.5155)</td>
<td>0.6428</td>
<td>0.4794</td>
</tr>
<tr>
<td>cos θ</td>
<td>0.8660</td>
<td>0.2588</td>
<td>0.3624</td>
<td>(–0.8569)</td>
<td>(–0.7660)</td>
<td>0.8776</td>
</tr>
<tr>
<td>tan θ</td>
<td>0.5774</td>
<td>(–3.7321)</td>
<td>2.5722</td>
<td>0.6016</td>
<td>(–0.8391)</td>
<td>0.5463</td>
</tr>
<tr>
<td>cosec θ</td>
<td>2.0000</td>
<td>(–1.0353)</td>
<td>1.0729</td>
<td>(–1.9398)</td>
<td>1.5557</td>
<td>2.0858</td>
</tr>
<tr>
<td>sec θ</td>
<td>1.1547</td>
<td>3.8637</td>
<td>2.7597</td>
<td>(–1.1670)</td>
<td>(–1.3054)</td>
<td>1.1395</td>
</tr>
<tr>
<td>cot θ</td>
<td>1.7321</td>
<td>(–0.2679)</td>
<td>0.3888</td>
<td>(1.6622)</td>
<td>(–1.1918)</td>
<td>1.8305</td>
</tr>
</tbody>
</table>

2. tan 53° = 1.33, cosec 53° = 1.25, sec 53° = 1.67, cot 53° = 0.75

3. a) 0.8529  
b) (–0.7711)  
c) (–1.7710)  
d) 0.2220  
e) 1.0321  
f) 1.4708

4. a) 0.3582  
b) 1.8508  
c) 16.5887  
d) 4.7046  
e) 2.4255  
f) 0.7071

Activity 5.10
1. \( \sin x = \frac{5}{13} \), tan \( x = \frac{5}{12} \), cot \( x = \frac{12}{5} \), sec \( x = \frac{13}{12} \), cosec \( x = \frac{13}{5} \)

2. \( \sin x = \frac{3}{\sqrt{34}} \), \( \cos x = \frac{5}{\sqrt{34}} \), cot \( x = \frac{5}{3} \), sec \( x = \frac{\sqrt{34}}{5} \), cosec \( x = \frac{\sqrt{34}}{3} \)

3. \( \sin x = \frac{2\sqrt{2}}{3} \), \( \cos x = \left(\frac{1}{3}\right) \), tan \( x = (–2\sqrt{2}) \), cot \( x = \left(\frac{1}{2\sqrt{2}}\right) \), cosec \( x = \frac{3}{2\sqrt{2}} \)

4. \( \sin x = \left(\frac{1}{2}\right) \), \( \cos x = \frac{\sqrt{3}}{2} \), tan \( x = \left(\frac{1}{\sqrt{3}}\right) \), cot \( x = (–\sqrt{3}) \), sec \( x = \frac{2}{\sqrt{3}} \)
Activity 5.11

1. a) 40°   b) 35°
    c) 50°   d) 72°
    e) 50°   f) 0.6238
    g) 0.7854 h) 1.0996

2. a) 71° 13' 37'' b) 62° 51' 38''
    c) 27° 28' 28'' d) 70° 11' 57''
    e) 58° 12' 04'' f) 75° 17' 25''

3. a) 1.2431 b) 0.0900
    c) 0.3340 d) 1.4243
    e) 0.8840 f) 1.0287

4. a) 59° 55' 42'' and 120° 04' 18'' b) 151° 13' 23'' and 208° 46' 37''
    c) 38° 06' 17'' d) 80° 47' 13''
    e) (−9° 01' 56'') and 9° 01' 56'' f) (−167° 32' 57''), (−12° 27' 03''),
       192° 27' 03'' and 347° 32' 57''

5. a) 1.413 and 1.728 b) 1.244 and 5.039
    c) 0.629 d) (−2.103) and (−1.038)
    e) 0.888 and 2.253 f) 0.656, 2.486, 3.797 and 5.628
Section 6 – Trigonometry part 2

6.1 Basic trigonometric identities

An identity is a mathematical term used to describe an equation which is true for all values of the variable involved. For example, the equation: \(3 + x = x + 3\) is a true statement no matter what values of \(x\) are used, so this equation is an identity.

There are many relationships between the six basic trigonometric quantities. These are expressed as trigonometric identities.

For example \(\frac{\sin A}{\cos A} = \tan A\) is a trigonometric statement that is true for all values of \(A\) so it is a trigonometric identity.

Below is a list of all the important trigonometric identities that you have already met. They have been established as definitions and the use of Pythagoras’ Theorem.

<table>
<thead>
<tr>
<th>Quotient identities</th>
<th>Reciprocal identities</th>
<th>Pythagorean identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{\sin A}{\cos A} = \tan A)</td>
<td>(\cosec A = \frac{1}{\sin A})</td>
<td>(\sin^2 A + \cos^2 A = 1)</td>
</tr>
<tr>
<td>(\frac{\cos A}{\sin A} = \cot A)</td>
<td>(\sec A = \frac{1}{\cos A})</td>
<td>(1 + \cot^2 A = \cosec^2 A)</td>
</tr>
<tr>
<td></td>
<td>(\cot A = \frac{1}{\tan A})</td>
<td>(1 + \tan^2 A = \sec^2 A)</td>
</tr>
</tbody>
</table>

Pythagorean identities are easily proved. Other, more complex identities, can also be proven. Use is generally made of algebraic techniques such as cross multiplication and factorisation.

A general approach is to start with the most complex side and work towards the simpler looking side. The side from which we start is designated as R.H.S. (Right Hand Side) or L.H.S. (Left Hand Side) whatever may be the case.

Use can always be made of the identities shown above.

The process is best illustrated by means of some examples.

**Example 1**  Prove that \((1 + \sec A)(1 - \cos A) = \sec A - \cos A\)

L.H.S. = \((1 + \sec A)(1 - \cos A)\)

= \(1 - \cos A + \sec A - \sec A \cos A\)

= \(\sec A - \cos A = R.H.S.\)
**Example 2** Prove that $\sec A - \cos A = \sin A \tan A$

L.H.S. = $\sec A - \cos A$

$$= \frac{1}{\cos A} - \cos A$$

$$= \frac{1 - \cos^2 A}{\cos A}$$

$$= \frac{\sin^2 A}{\cos A}$$

$$= \sin A \times \frac{\sin A}{\cos A}$$

$$= \sin A \tan A = \text{R.H.S.}$$

Sometimes, when proving an identity, both the denominator and numerator need to be multiplied by an expression. Frequently use is made of the fact that:

$$\sin^2 A = 1 - \cos^2 A \text{ or } \cos^2 A = 1 - \sin^2 A.$$

A typical example follows.

**Example 3** Prove that $\frac{\cos A}{1 + \sin A} = \sec A (1 - \sin A)$

L.H.S. = $\frac{\cos A}{1 + \sin A} \times \frac{1 - \sin A}{1 - \sin A}$

$$= \frac{\cos A (1 - \sin A)}{1 - \sin^2 A}$$

$$= \frac{\cos A (1 - \sin A)}{\cos^2 A}$$

$$= \frac{1 - \sin A}{\cos A}$$

$$= \sec A (1 - \sin A) = \text{R.H.S.}$$
Activity 6.1
In the following problems show that the identities are true for all values of $\theta$.

1. \[ \frac{2}{1 - \cos \theta} = \frac{2\sec \theta}{\sec \theta - 1} \]

2. \[ \frac{\sin^2 \theta}{1 + \cos \theta} = 1 - \cos \theta \]

3. \[ \frac{1 - \cos^2 \theta}{\sec^2 \theta - 1} = \cos^2 \theta \]

4. \[ \frac{1 + \cot^2 \theta}{1 + \tan^2 \theta} = \cot^2 \theta \]

5. \[ (\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta \]

6. \[ \frac{\sin^2 \theta + \cot^2 \theta}{\csc^2 \theta - \cos^2 \theta} = 1 \]

7. \[ \csc^2 \theta + \tan^2 \theta = \cot^2 \theta + \sec^2 \theta \]

8. \[ \csc^4 \theta (\sin^2 \theta - \sin^2 \theta \cos^2 \theta) = 1 \]

6.2 Identities derived from the unit circle
In this topic we consider various properties of the unit circle which lead to important sets of identities which in turn can be used to solve equations and draw graphs, material which we shall study later in this section. We list the identities in terms of radians but the identities could, of course, also have been stated in terms of degrees. We restrict the identities to sine and cosine because the corresponding identities for tangent can be easily derived using the fact that $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

1. Identities involving the complement
In the diagram below, if angle AOP ($= t$) is associated with $P(a, b)$, then angle AOQ ($= \frac{\pi}{2} - t$) is associated with $Q(b, a)$.

![Diagram of the unit circle and angles](image)

Hence $\sin \left( \frac{\pi}{2} - t \right) = \cos t$ ($= a$ in the diagram) and $\cos \left( \frac{\pi}{2} - t \right) = \sin t$ ($= b$ in the diagram).
In this following diagram, if angle AOP ($= t$) is associated with $P(a, b)$, then angle AOQ ($= \frac{\pi}{2} + t$) is associated with $Q(-b, a)$.

Hence $\sin \left( \frac{\pi}{2} + t \right) = \cos t$ ($= a$ in the diagram) and $\cos \left( \frac{\pi}{2} + t \right) = -\sin t$ ($= -b$ in the diagram).

In summary:

<table>
<thead>
<tr>
<th>$\sin \left( \frac{\pi}{2} - t \right)$</th>
<th>$\sin \left( \frac{\pi}{2} + t \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos t$</td>
<td>$\cos t$</td>
</tr>
<tr>
<td>$\cos \left( \frac{\pi}{2} - t \right)$</td>
<td>$\cos \left( \frac{\pi}{2} + t \right)$</td>
</tr>
<tr>
<td>$\sin t$</td>
<td>$-\sin t$</td>
</tr>
</tbody>
</table>

2. **Identities involving the supplement**

Using a similar method to the one just shown we can establish the following identities:

<table>
<thead>
<tr>
<th>$\sin(\pi - t)$</th>
<th>$\sin(\pi + t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin t$</td>
<td>$-\sin t$</td>
</tr>
<tr>
<td>$\cos(\pi - t)$</td>
<td>$-\cos t$</td>
</tr>
<tr>
<td>$\cos(\pi + t)$</td>
<td>$-\cos t$</td>
</tr>
</tbody>
</table>

3. **Identities for negative angles**

The terminal rays of angle of size ($-t$) and that of ($2\pi - t$) are identical. Hence:

<table>
<thead>
<tr>
<th>$\sin (-t)$</th>
<th>$\sin (2\pi - t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\sin t$</td>
<td>$-\sin t$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\cos(-t)$</th>
<th>$\cos (2\pi - t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos t$</td>
<td>$\cos t$</td>
</tr>
</tbody>
</table>
Example 1  If the value of \( \sin 30^\circ = 0.5 \), what is the value of \( \sin 150^\circ \)?

Since \( \sin (180^\circ - t) = \sin t \),
\[
\sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = 0.5
\]

Example 2  Express \( \cos \left( -\frac{2\pi}{3} \right) \) in terms of an acute angle.

\[
\cos \left( -\frac{2\pi}{3} \right) = \cos \left( \frac{2\pi}{3} \right) = \cos \left( \pi - \frac{\pi}{3} \right) = -\cos \frac{\pi}{3}
\]

Example 3  Express the sine, cosine and tangent of \( 120^\circ \) in terms of an acute angle.

\[
\sin 120^\circ = \sin (180^\circ - 60^\circ) = \sin 60^\circ \\
\cos 120^\circ = \cos (180^\circ - 60^\circ) = -\cos 60^\circ \\
\tan 120^\circ = \tan (180^\circ - 60^\circ) = -\tan 60^\circ
\]

Example 4  Express the sine, cosine and tangent of \( 390^\circ \) in terms of an acute angle.

The terminal ray of an angle of \( 390^\circ \) falls in quadrant 1 as shown below.

\[
\sin 390^\circ = \sin (360^\circ + 30^\circ) = \sin 30^\circ \\
\cos 390^\circ = \cos (360^\circ + 30^\circ) = \cos 30^\circ \\
\tan 390^\circ = \tan (360^\circ + 30^\circ) = \tan 30^\circ
\]
Example 5  Express \( \cos \left(t - \frac{\pi}{2}\right) \) in terms of \( t \).

\[
\cos \left(t - \frac{\pi}{2}\right) = \cos \left[ -\left(\frac{\pi}{2} - t\right)\right]
\]

Let \( x = \frac{\pi}{2} - t \) then \( \cos \left[ -\left(\frac{\pi}{2} - t\right)\right] = \cos (-x) = \cos x = \cos \left(\frac{\pi}{2} - t\right) = \sin t
\]

Activity 6.2

1. Express the following in terms of the trigonometric ratios of \( x \).
   
   a) \( \tan \left(\frac{\pi}{2} - x\right) \)  
   b) \( \cosec \left(\pi - x\right) \)

   c) \( \sec \left(\frac{\pi}{2} + x\right) \)  
   d) \( \cot \left(\frac{\pi}{2} + x\right) \)

   e) \( \sec (-x) \)  
   f) \( \cosec \left(\pi + x\right) \)

   g) \( \cos \left(\frac{3\pi}{2} - x\right) \)  
   h) \( \sin \left(\frac{\pi}{2} + x\right) \)

   i) \( \tan (-x) \)  
   j) \( \sin \left(x - \frac{\pi}{2}\right) \)

2. Express each of the following in terms of an acute angle.
   
   a) \( \sin 135^\circ, \cos 135^\circ \)  
   b) \( \cos 210^\circ, \tan 210^\circ \)

   c) \( \cos 300^\circ, \tan 300^\circ \)  
   d) \( \sin 415^\circ, \tan 415^\circ \)

6.3  Trigonometric formulae

In this topic we will develop four sets of trigonometric identities, generally called formulae, that are widely used in higher level Mathematics subjects such as Calculus. You don’t have to memorise the formulae but you should gain enough practice in using them that you at least recognise the formulae.

We start with the addition formulae.

It is clear that \( \sin A + \sin B \neq \sin (A + B) \). All that we have to do to satisfy ourselves of this is to put \( A = 20^\circ \) and \( B = 30^\circ \). Then, working with 4 dp:

\[
\sin 20^\circ + \sin 30^\circ = 0.3420 + 0.5000 = 0.8420
\]

where as: \( \sin 50^\circ = 0.7660 \)

We can develop formulae for \( \sin (A \pm B) \), \( \cos (A \pm B) \) and \( \tan (A \pm B) \) in terms of \( A \) and they are called the addition formulae.
1. The addition formulae

To establish the addition formulae we proceed as follows using the definitions of \( \sin \) and \( \cos \) in the unit circle.

![Unit Circle Diagram]

Notice that \( \angle QOR \) is \( A \) while \( \angle POR \) is \( B \) making \( \angle QOP = A - B \).

The distance \( QP \) can be found using the distance between two points formula and the cosine rule in \( \triangle QOP \). Since it is the same distance no matter what method is used we obtain the equation:

\[
(cos A - cos B)^2 + (sin A - sin B)^2 = 1^2 + 1^2 - 2(1)(1)cos(A - B)
\]

By manipulating this equation we obtain a formula for \( cos(A - B) \) thus:

\[
\begin{align*}
\cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B &= 1 + 1 - 2 \cos(A - B) \\
(\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B) &= 2 - 2 \cos(A - B) \\
1 + 1 - 2(\cos A \cos B + \sin A \sin B) &= 2 - 2 \cos(A - B)
\end{align*}
\]

Hence:

\[
\cos(A - B) = \cos A \cos B + \sin A \sin B
\]

By using identities we established earlier and using various substitutions, another three similar identities can be established.

These four addition formulae are summarised as:

\[
\begin{align*}
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B
\end{align*}
\]
Example 1  Evaluate, using 3 dp, \(\cos (A - B)\) and \(\cos A - \cos B\) when \(A = 60^\circ\) and \(B = 30^\circ\).

\[
\begin{align*}
\cos A - \cos B &= \cos 60^\circ - \cos 30^\circ \\
&= 0.500 - 0.866 \\
&= -0.366
\end{align*}
\]

\[
\begin{align*}
\cos (A - B) &= \cos (60^\circ - 30^\circ) \\
&= \cos 30^\circ \\
&= 0.866
\end{align*}
\]

Using the formula:

\[
\begin{align*}
\cos (A - B) &= \cos A \cos B + \sin A \sin B \\
\cos (60^\circ - 30^\circ) &= \cos 60^\circ \cos 30^\circ + \sin 60^\circ \sin 30^\circ \\
&= (0.500)(0.866) + (0.866)(0.500) \\
&= 0.433 + 0.433 \\
&= 0.866
\end{align*}
\]

Example 2  Simplify: \(\sin t \cos 70^\circ - \cos t \sin 70^\circ\).

Using the formula \(\sin (A - B) = \sin A \cos B - \cos A \sin B\) this becomes: \(\sin (t - 70^\circ)\).

Many of the identities established earlier from the unit circle can also be derived using the addition formulae.

Example 3  Develop a formula for \((180^\circ + \theta)\).

\[
\begin{align*}
\cos (A + B) &= \cos A \cos B - \sin A \sin B \\
\cos (180^\circ + \theta) &= \cos 180^\circ \cos \theta - \sin 180^\circ \sin \theta
\end{align*}
\]

but \(\sin 180^\circ = 0\) and \(\cos 180^\circ = -1\)

Hence: \(\cos (180^\circ + \theta) = -\cos \theta\)

Example 4  Show that \(\sin (90^\circ + \theta) = \sin (90^\circ - \theta)\).

\[
\begin{align*}
\sin (90^\circ - \theta) &= \cos \theta \\
\sin (90^\circ + \theta) &= \sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta \\
\text{but } \sin 90^\circ &= 1 \text{ and } \cos 90^\circ = 0
\end{align*}
\]

so \(\sin (90^\circ + \theta) = \cos \theta = \sin (90^\circ - \theta)\)

Of course, more complicated identities can now be derived or proven.
Example 5

Establish the identity: \[
\frac{\sin(A + B)}{\cos A \sin B} = \tan A \cot B + 1
\]

\[
\frac{\sin(A + B)}{\cos A \sin B} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \sin B} = \tan A \cot B + 1
\]

2. The double angle formulae

By putting \( B = A \) in the addition formulae we obtain the **double angle** formula.

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]

becomes: \[
\sin(A + A) = \sin A \cos A + \cos A \sin A
\]

or \[
\sin 2A = 2 \sin A \cos A
\]

If we put \( B = A \) in the formula for \((A + B)\) we obtain:

\[
\cos 2A = \cos^2 A - \sin^2 A
\]

Using \( \cos^2 A + \sin^2 A = 1 \) we can develop some other useful formulae.

By adding \( \cos^2 A - \sin^2 A = \cos 2A \) to \( \cos^2 A + \sin^2 A = 1 \), we obtain:

\[
2 \cos^2 A = 1 + \cos 2A
\]

which gives: \[
\cos 2A = 2 \cos^2 A - 1
\]

By subtracting \( \cos^2 A - \sin^2 A = \cos 2A \) from \( \cos^2 A + \sin^2 A = 1 \), we obtain:

\[
2 \sin^2 A = 1 - \cos 2A
\]

which gives: \[
\cos 2A = 1 - 2 \sin^2 A
\]

Example 6

Expand: \( \sin 8X \)

\[
\sin 8X = 2 \sin 4X \cos 4X
\]

Example 7

Prove the identity: \( \sin^2(X + 90^\circ) + \sin^2(X - 90^\circ) = 1 + \cos 2X \)

L.H.S. = \( \sin^2(X + 90^\circ) + \sin^2(X - 90^\circ) \)

\[
= \left[ \sin(90 + X) \right]^2 + \left[ -\sin(90 - X) \right]^2
\]

\[
= \cos^2 X + \cos^2 X
\]

\[
= 2 \cos^2 X
\]

\[
= 1 + \cos 2X
\]

= R.H.S.
Section 6 Trigonometry part 2

Example 8 If \( \cos 2X = \frac{1}{8} \), calculate the values of \( \sin X \).

\[
\sin X = \pm \sqrt{\frac{1 - \cos 2X}{2}} = \pm \sqrt{\frac{1 - \frac{1}{8}}{2}}
\]

\[
= \pm \sqrt{\frac{7}{16}} = \pm \frac{\sqrt{7}}{4}
\]

3. The sum-to-product formulae

By algebraically manipulating the addition formulae we can derive some formulae for writing the sum (or difference) between two trigonometric quantities, as a product between two other trigonometric quantities. This procedure is similar to factorising and is useful in manipulating and simplifying trigonometric expressions.

Consider the addition formulae:

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B
\]

\[
\sin(A - B) = \sin A \cos B - \cos A \sin B
\]

Adding these two equations gives:

\[
\sin(A + B) + \sin(A - B) = 2 \sin A \cos B
\]

Now, using \( A + B = S \) and \( A - B = T \)

addition gives: \( 2A = S + T \) or \( A = \frac{S + T}{2} \)

and subtraction gives: \( 2B = S - T \) or \( B = \frac{S - T}{2} \)

Now, rewriting \( \sin(A + B) + \sin(A - B) = 2 \sin A \cos B \) in terms of \( S \) and \( T \) we obtain:

\[
\sin S + \sin T = 2 \sin \left( \frac{S + T}{2} \right) \cos \left( \frac{S - T}{2} \right)
\]

This is the first of the sum-to-product formulae. Note that the expression on the left-hand side is a sum between \( \sin S \) and \( \sin T \), while the right-hand side is a product between \( 2 \sin \left( \frac{S + T}{2} \right) \) and \( \cos \left( \frac{S - T}{2} \right) \).
Example 9  Write $\sin 100^\circ + \sin 80^\circ$ as a product.

$$\sin 100^\circ + \sin 80^\circ = 2 \sin \left( \frac{100 + 80}{2} \right) \cos \left( \frac{100 - 80}{2} \right)$$

$$= 2 \sin 90^\circ \cos 10^\circ$$

$$= 2 \cos 10^\circ$$

Notice that sometimes the formula allows some simplification of a result. Check with a calculator that $\sin 100^\circ + \sin 80^\circ = 2 \cos 10^\circ$.

Three other sum-to-product formulae can be obtained, by similar methods.

They are:  

$$\sin S - \sin T = 2 \cos \left( \frac{S + T}{2} \right) \sin \left( \frac{S - T}{2} \right)$$

$$\cos S + \cos T = 2 \cos \left( \frac{S + T}{2} \right) \cos \left( \frac{S - T}{2} \right)$$

$$\cos T - \cos S = 2 \sin \left( \frac{S + T}{2} \right) \sin \left( \frac{S - T}{2} \right)$$

Take care with the last of these. Notice that it is slightly different in form to the others. 

The first term on the left is the cosine of $T$ not $S$.

The order of subtraction is important and writing this formula this way is the tidiest way of expressing it.

Example 10  Show that: 

$$\frac{\cos 2X + \cos X}{\sin 2X - \sin X} = \cot \frac{X}{2}$$

L.H.S. = 

$$\frac{\cos 2X + \cos X}{\sin 2X - \sin X}$$

$$= \frac{2 \cdot \cos \left( \frac{2X + X}{2} \right) \cos \left( \frac{2X - X}{2} \right)}{2 \cdot \cos \left( \frac{2X + X}{2} \right) \sin \left( \frac{2X - X}{2} \right)}$$

$$= \frac{2 \cdot \cos \frac{3X}{2} \cos \frac{X}{2}}{2 \cdot \cos \frac{3X}{2} \sin \frac{X}{2}}$$

$$= \frac{\cos \frac{X}{2}}{\sin \frac{X}{2}}$$

$$= \cot \frac{X}{2} = \text{R.H.S.}$$

Changing sums with products enabled us to cancel factors in this last example. In fact, factoring is an important role for these formulae.
Example 11  Factor: \( \sin 3A - \sin 5A + \sin 2A \).

Group the first two terms together and apply the sum-to-product formula.

\[
\begin{align*}
(\sin 3A - \sin 5A) + \sin 2A &= 2 \cos \left( \frac{3A + 5A}{2} \right) \sin \left( \frac{3A - 5A}{2} \right) + \sin 2A \\
&= -2 \cos 4A \sin A + \sin 2A \\
&= -2 \cos 4A \sin A + 2 \sin A \cos A \\
&= 2 \sin A (\cos 4A - \cos A)
\end{align*}
\]

where we used \( \sin (-A) = -\sin A \) and have written \( \sin 2A \) in terms of the single angle \( A \) so we can get a common factor.

Now apply the sum-to-product formula to the expression in the brackets:

\[
= 2 \sin A \cdot \left( 2 \cdot \sin \frac{4A + A}{2} \sin \frac{4A - A}{2} \right)
= 2 \sin A \cdot 2 \cdot \sin \frac{5A}{2} \sin \frac{3A}{2}
= 4 \sin A \cdot \sin \frac{5A}{2} \sin \frac{3A}{2}
\]

4. The product-to-sum formulae

The formula \( \sin (A + B) + \sin (A - B) = 2 \sin A \cos B \) which we established when deriving the sum-to-product formulae can be used to write a product as a sum.

In fact

\[
2 \sin A \cos B = \sin (A + B) + \sin (A - B)
\]

In a similar way, other product-to-sum formulae are obtained. They are:

\[
\begin{align*}
2 \cos A \sin B &= \sin (A + B) - \sin (A - B) \\
2 \cos A \cos B &= \cos (A + B) + \cos (A - B) \\
2 \sin A \sin B &= \cos (A - B) - \cos (A + B)
\end{align*}
\]

The following examples illustrate the use of these formulae.

Example 12  Express \( \cos 80^\circ \cos 65^\circ \) as a sum or difference with acute angles greater than \( 45^\circ \).

\[
\begin{align*}
\cos 80^\circ \cos 65^\circ &= \frac{1}{2} [\cos (80 + 65)^\circ + \cos (80 - 65)^\circ] \\
&= \frac{1}{2} (\cos 145^\circ + \cos 15^\circ)
\end{align*}
\]
Now \( \cos 145^\circ = \cos (90 + 55)^\circ \)
\[ = \cos 90^\circ \cos 55^\circ - \sin 90^\circ \sin 55^\circ \]
\[ = -\sin 55^\circ \]
and \( \cos 15^\circ = \cos (90 - 75)^\circ \)
\[ = \cos 90^\circ \cos 75^\circ + \sin 90^\circ \sin 75^\circ \]
\[ = \sin 75^\circ \]

So \( \cos 80^\circ \cos 65^\circ = \frac{1}{2} (\sin 75^\circ - \sin 55^\circ) \)

**Example 13** Show that \( \cos 6 X \cos X + \sin 4 X \sin X = \frac{1}{2} (\cos 7 X + \cos 3 X) \).

L.H.S. = \( \cos 6 X \cos X + \sin 4 X \sin X \)
\[ = \frac{1}{2} (\cos 7 X + \cos 5 X) + \frac{1}{2} (\cos 3 X - \cos 5 X) \]
\[ = \frac{1}{2} (\cos 7 X + \cos 5 X + \cos 3 X - \cos 5 X) \]
\[ = \frac{1}{2} (\cos 7 X + \cos 3 X) = \text{R.H.S.} \)

**Activity 6.3**

1. If \( \sin A = \frac{3}{5} \) and \( \cos B = \frac{5}{13} \) and \( A \) and \( B \) are acute, evaluate:
   a) \( \cos (A + B) \)
   b) \( \tan (A - B) \)

2. Given \( 0 \leq x \leq \frac{\pi}{2} \) and \( \frac{\pi}{2} \leq y \leq \pi \) and \( \tan x = \frac{1}{8} \) and \( \tan y = \left( -\frac{4}{7} \right) \) evaluate:
   a) \( \sin (x + y) \)
   b) \( \cos (x + y) \)

3. Find \( \cos 2x \) given:
   a) \( \cos x = \frac{2}{3} \)
   b) \( \sin x = \frac{1}{4} \)

4. Simplify: \( \cos 27^\circ \sin 18^\circ + \sin 27^\circ \cos 18^\circ \)

5. Simplify:
   a) \( \sin (20^\circ + x) \cos (20^\circ + x) \)
   b) \( \sin^2 3x - \cos^2 3x \)
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6. Simplify: \((\cos A - \sin A)^2\).

7. Without using a calculator show that \(\sin 10^\circ + \sin 130^\circ - \sin 70^\circ = 0\).

8. Express \(2 \cos 68^\circ \sin 42^\circ\) as a sum or difference of sines and cosines with acute angles > 45°.

9. Express \(\sin 168^\circ - \sin 254^\circ\) as a product with acute angles, less than 45°.

10. Show that \(2 \cos (45^\circ + X) \cos (135^\circ - X) = \sin 2X - 1\).

11. Express \(4 \cos X \cos 2X \sin 4X\) as the sum of four sines.

12. Express in the form \(R \cos (\varphi \pm \alpha)\) the sum of voltages:

\[
V_1 = 2 \sin \left(\varphi + \frac{\pi}{4}\right) \text{ volts and } V_2 = 3 \cos \left(\varphi - \frac{\pi}{6}\right) \text{ volts.}
\]

6.4 The sine and cosine graphs

When we consider the definition of sine and cosine, it is clear that when a rotational angle is increased by \(2\pi\) its trigonometric values are unchanged because the position of the terminal ray is identical to the original angle. That is, the values of \(\sin x\) and \(\cos x\) repeat at each stage that \(x\) is increased by \(2\pi\). For this reason we say that sine and cosine are periodic with period \(2\pi\).

To draw one period of \(y = \sin x\) or \(y = \cos x\), we could start in the first quadrant by graphing the sin and cos values of angles for which we know the exact values. Following that we could make use of our knowledge of the signs of the sine and cosine functions in the other quadrants in combination with the identities applicable to the unit circle to complete the full period.

However, here we have used the computer drawing package GRAPHMATICA. Of course, a graphics calculator could also have been used.

We start by drawing the basic sine and cosine graphs. Note that the scale on the \(x\) axis is drawn in terms of \(\pi\). We could have used a scale marked in degrees but in more advanced applications of this type of trigonometry (e.g., electronic engineering), radians are more useful.
Use quadratic, exponential, logarithmic and trigonometric functions and matrices.

Graph of $y = \sin x$

Graph of $y = \cos x$
In the examples to follow, we shall concentrate on how the graph of \( y = \sin x \) can be altered, but the same principles apply to the graph of \( y = \cos x \).

By definition, the **amplitude** of a trigonometric graph is defined as half the distance between the minimum and maximum values. Notice that both graphs have a minimum value of \((-1)\) and a maximum value of 1 and hence their amplitude is 1.

It is possible to change both the period and amplitude of the sine and cosine graphs. In fact, if:

\[
y = a \sin bx \text{ or } y = a \cos bx
\]

then the amplitude is \( |a| \) and the period is \( \frac{2\pi}{|b|} \) where the symbol \( | \ | \) means the **absolute value** (the number regardless of sign).

**Example 1**  What is the period and amplitude of \( y = -3 \sin 2x \)?

Here \( a = (-3) \) and \( b = 2 \).

Hence the amplitude is \( |-3| = 3 \) and the period is \( \frac{2\pi}{2} = \pi \).

Knowledge of the amplitude and period facilitates the graphing process.

**Example 2**  On the same graph paper over the domain \( 0 \leq x \leq 2\pi \) draw the graphs of \( y = \sin x \) and \( y = 2\sin x \).
Example 3 On the same graph paper over the domain $0 \leq x \leq 2\pi$ draw the graphs of $y = \sin x$ and $y = \sin 2x$.

The graph can also be inverted.

Example 4 On the same graph paper over the domain $0 \leq x \leq 2\pi$ draw the graphs of $y = \sin x$ and $y = -\sin x$. 
Trigonometric graphs can also be shifted vertically or horizontally. In fact, if
\[ y = a \sin (bx + c) + d \] or \[ y = a \cos (bx + c) + d \]
then the vertical displacement is \( d \) and the horizontal displacement is \( \cfrac{-c}{|b|} \).
The horizontal displacement is called the **phase shift**.

**Example 5**  On the same graph paper over the domain \( 0 \leq x \leq 2\pi \) draw the graphs of
\( y = \sin x \) and \( y = \sin x + 1 \).

For \( y = \sin(x + 1) \), the value of \( c = 1 \) and consequently, the phase shift is \( \cfrac{-1}{1} = -1 \).
That is, the graph of \( y = \sin x \) has been shifted 1 unit (1 radian) to the left.
We finish with a complicated example which incorporates all of the concepts illustrated above.

**Example 7**  
Draw \( y = -0.5 \sin \left(2x - \frac{\pi}{4}\right) - 0.1\) over the domain \( 0 \leq x \leq 2\pi\).

Here \( a = (-0.5), b = 2, c = \left(\frac{\pi}{4}\right)\) and \( d = (-0.1)\).

Hence the amplitude is \(|-0.5| = 0.5\), the period is \(\frac{2\pi}{2} = \pi\)  
(there will be two cycles!), the vertical shift is 0.1 down and the phase shift is \(\left(-\frac{\pi/4}{2}\right) = \frac{\pi}{8} = 0.125\pi\).

All this results in the following graph.

![Graph of y = -0.5 sin(2x - pi/4) - 0.1](image)

**Activity 6.4**

1. Sketch the graphs of each of the following trigonometric functions for \(0^\circ \leq x \leq 360^\circ\).

   a) \( y = 2\sin x \)
   
   b) \( y = 2\cos x \)
   
   c) \( y = 3\cos 2x \)
   
   d) \( y = \cos (x - 115^\circ) \)
   
   e) \( y = -\sin (3x - 240^\circ) \)
   
   f) \( y = 1.25\sin (2.5x + 270^\circ) \)
2. Sketch the graphs of each of the following trigonometric functions over \(0 \leq x \leq 2\pi\).

a) \(y = 2\cos x\)  
b) \(y = 0.5 \sin 3x\)  
c) \(y = -2\cos x\)  
d) \(y = 1 - \sin x\)  
e) \(y = 3 + \sin\frac{1}{2}x\)  
f) \(y = \frac{1}{2} \sin (2x + \pi)\)

### 6.5 Solving trigonometric equations using graphs

As shown, trigonometric equations can be solved using the unit circle. A more universal and convenient method, especially if a graphics calculator is available, is to use trigonometric graphs.

As usual, we shall illustrate this process by means of some examples.

**Example 1** Solve \(\sin x = 0.8\) for \(0 \leq x \leq 2\pi\) correct to 4 dp.

We draw \(y = \sin x\) over the domain \(0 \leq x \leq 2\pi\) and on the same graph draw \(y = 0.8\). The \(x\) values of the points of intersection will be our solutions.

Here \(x = 0.9273\) and \(x = 2.2143\) which we obtained from a calculator.
Example 2  Solve $2\cos x = -0.5471$ for $-\pi \leq x \leq 2\pi$ correct to 4 dp.

When using the unit circle, we should write the equation as

$$\cos x = \frac{-0.5471}{2} = -0.27355.$$  

From this the reference angle $\alpha$ can be obtained by calculating $\alpha = \cos^{-1}(0.27355) = 1.2937$ (4 dp).

Then realising that cosine is negative in quadrants 2 and 3, and taking into account the domain, the three solutions can be obtained.

In fact, $x$ is equal to:

$$-\pi + 1.2937 = -1.8479, \pi - 1.2937 = 1.8479 \text{ or } \pi + 1.2937 = 4.4353$$

The graphical method involves drawing $y = 2\cos x$ and $y = (-0.5471)$ from which the three solutions can be read off.
Example 3  Solve \( 2 \cos \left( \frac{\pi x}{2} + 1 \right) = 0.5 \) for \( 0 \leq x \leq 4 \) correct to 1 decimal place.

Sketch \( y = 2 \cos \left( \frac{\pi x}{2} + 1 \right) \) and \( y = 0.5 \) on the one set of axes.

(You should calculate the period, the amplitude and the phase shift of the cosine graph.)

From the graph, the solutions are \( x = 0.2 \) and \( x = 2.5 \).

Activity 6.5

1. Draw the graph of \( y = \cos x \) and solve, correct to the nearest degree, \( \cos x = (-0.7) \) for \( 0^\circ \leq x \leq 360^\circ \).

2. Draw the graph of \( y = -\sin x \) and solve, correct to 4 dp, the equation \( -\sin x = 0.6563 \) for \( -\pi \leq x \leq \pi \).

3. Draw the graph of \( y = 3\cos 2x \) between \( 0^\circ \) and \( 180^\circ \) and hence solve to the nearest degree.
   a) \( 3\cos 2x = 2 \)
   b) \( 3\cos 2x = (-1) \)

4. Sketch the graph of the function defined by \( y = 2 \sin \left( x + \frac{4\pi}{9} \right) \) for \( 0 \leq x \leq 2\pi \).

   Use this graph to solve the equation \( 2 \sin \left( x + \frac{4\pi}{9} \right) = \left( -\frac{1}{2} \right) \) for \( 0 \leq x \leq 2\pi \), correct to 4 dp.

5. Solve \( 5 \cos \left( 2x - \frac{\pi}{4} \right) = 4 \) for \( 0 \leq x \leq 5 \), correct to 4 dp.
6.6 The tangent function

The appearance of this graph $y = \tan x$ is very different from that of the sine and cosine curves.

Graph of $y = \tan x$

Note that the function is periodic and goes through a cycle every $\pi$ units. So the period is $\pi$.

Asymptotes occur at $x = 0.5\pi$ (ie $\frac{\pi}{2}$) and $x = 1.5\pi$ (ie $\frac{3\pi}{2}$). Sometimes these lines are included in the graph as in the following diagram.

The graph of $y = a \tan x$ has the same shape and period but as $a$ increases so does the steepness of the graph segments. The tangent graph does not have an amplitude.
Example 1 Sketch the graph of \( y = \frac{1}{2} \tan x \) and \( y = 2 \tan x \) for \( 0 \leq x \leq \frac{\pi}{2} \).

The graph of \( y = \tan bx \) has period \( \frac{\pi}{|b|} \) since it repeats its cycle \( b \) times in an interval of size \( \pi \).

Example 2 Sketch one cycle of \( y = \tan 2x \). Determine the period.

The period is \( \frac{\pi}{2} \).
Equations, can of course, also be solved.

**Example 3** Solve \( 1 - 2\tan 3x = (-4.876) \) for \( 0 \leq x \leq 0.5\pi \), correct to 2 dp.

\[
\begin{align*}
\text{The solutions are } x &= 0.41 \text{ and } x = 1.46.
\end{align*}
\]

**Activity 6.6**

1. Draw one cycle of \( y = -\tan x \). Use degree measure.

2. Graph \( y = 3\tan\left(\frac{1}{2}x\right) \) for \( 0^\circ \leq x \leq 360^\circ \)

3. Sketch the graph of the function defined by: \( y = \frac{1}{2}\tan 2x \) for \( -\frac{3\pi}{4} \leq x \leq \frac{3\pi}{4} \)

   Hence solve the equation \( \frac{1}{2}\tan 2x = (-13) \) in this interval, correct to 4 dp.
Assessment 6

1. Prove the identity \( \frac{\cos^2 \theta}{1 + \sin \theta} = 1 - \sin \theta \).

2. Express \( \sin 225°, \cos 225° \) in terms of an acute angle.

3. Given \( \sin x = 0.8 \), find \( \sin 2x \).

4. Use the addition formulae to show that:
   \[
   \cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B.
   \]

5. Show that \( \cos(45° - A) \cos(45° + A) = \frac{1}{2} \cos 2A \).

6. If \( \cos 2x = \frac{7}{25} \) and \( 135° < x < 180° \), calculate the value of \( \cos x \).

7. Show that \( \sin B + \cos A \sin (A - B) = \sin A \cos (A - B) \).

8. Show that \( \frac{\sin 3X + \sin 4X}{\cos 3X + \cos 5X} = \tan 4X \).

9. Use sum-to-product formulae to show, without a calculator, that:
   \( \cos 25° - \cos 35° - \cos 85° = 0 \).

10. Show, using the product sum-to-formulae that:
    \( \sin 4x \cos x + \cos 5x \sin 2x = \sin 6x \cos x \).

11. Factorise \( \sin x + \sin 2x + \sin 3x \).

12. Draw a sketch of \( y = 2\sin 0.5x \) for \( 0 \leq x \leq 2\pi \).

13. Draw an accurate graph of \( y = 2\cos 3x \) for \( 0 \leq x \leq \pi \), and use it to solve the equation \( 2\cos 3x = -1 \) over this domain.

14. What are the amplitude, period and phase shift of \( y = 2\cos \left( 2x + \frac{\pi}{4} \right) \)?

15. Draw one period of \( y = \tan \left( \frac{1}{3} x \right) \).

16. Graphically solve the equation \( 2\sin x = \sin 2x \) for values of \( x \) between 0 and \( 2\pi \).
   Give the exact solution(s) in terms of \( \pi \).
Answers to activities

**Activity 6.1**

Proofs should show that the identities are true.

**Activity 6.2**

1. a) \( \cot x \)  
   b) \( \cosec x \)  
   c) \(-\cosec x\)  
   d) \(-\tan x\)  
   e) \( \sec x \)  
   f) \(-\cosec x\)  
   g) \(-\sin x\)  
   h) \( \cos x \)  
   i) \(-\tan x\)  
   j) \(-\cos x\)

2. a) \( \sin 45^\circ, -\cos 45^\circ \)  
   b) \(-\cos 30^\circ, \tan 30^\circ\)  
   c) \(\cos 60^\circ, -\tan 60^\circ\)  
   d) \(\sin 55^\circ, \tan 55^\circ\)

**Activity 6.3**

1. a) \( \begin{pmatrix} -16 \\ 65 \end{pmatrix} \)  
   b) \( \begin{pmatrix} -33 \\ 65 \end{pmatrix} \)

2. a) \( \frac{5}{13} \)  
   b) \( \begin{pmatrix} -12 \\ 13 \end{pmatrix} \)

3. a) \( \begin{pmatrix} -1 \\ 9 \end{pmatrix} \)  
   b) \( \frac{7}{8} \)

4. \( \frac{1}{\sqrt{2}} \)

5. a) \( \frac{1}{2} \sin(40^\circ + 2x) \)  
   b) \(-\cos 6x\)

6. \(1 - \sin 2x\)

7. Factorise using the sum-to-product formulae.

8. \( \sin 70^\circ - \cos 64^\circ \)

9. \(-2\cos 31^\circ \sin 43^\circ\)

10. Proof should show that it is true.

11. \( \sin x + \sin 3x + \sin 5x + \sin 7x \).

12. \(4.96 \cos(\phi - 36^\circ) \text{ volts or } 4.96 \cos(\phi - 0.6283)\).
Activity 6.4

1. a) ![Graph a)

   ![Graph b)

   c) ![Graph c)

   d) ![Graph d)

   e) ![Graph e)

   f) ![Graph f)

2. a) ![Graph a)

   b) ![Graph b)
Use quadratic, exponential, logarithmic and trigonometric functions and matrices.
Activity 6.5

1. \(x = 134^\circ\) and \(226^\circ\)

2. \(x = -2.4257\) and \(-0.7159\)
3. a) $x = 24^\circ$ and $156^\circ$

b) $x = 55^\circ$ and $125^\circ$
4.

\[ x = 1.9980 \text{ and } 4.6342 \]

5.

\[ x = 0.0709, 0.7144, 3.2125 \text{ and } 3.8560 \]
**Activity 6.6**

1. Use quadratic, exponential, logarithmic and trigonometric functions and matrices

2. EDX140

3. $x = (-2.3370), (-0.7662) \text{ and } 0.8046$

---

Use quadratic, exponential, logarithmic and trigonometric functions and matrices
Section 7 – Matrices

7.1 Definitions

Even if you have never heard of matrices the chances are that you will have looked at them frequently. A calendar, a premiership table, a shopping list, a table of test scores are all examples of matrices.

Consider, for example, the following table of test scores:

<table>
<thead>
<tr>
<th></th>
<th>Test 1</th>
<th>Test 2</th>
<th>Test 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biology</td>
<td>45</td>
<td>59</td>
<td>89</td>
</tr>
<tr>
<td>Chemistry</td>
<td>35</td>
<td>54</td>
<td>83</td>
</tr>
<tr>
<td>Mathematics</td>
<td>65</td>
<td>65</td>
<td>75</td>
</tr>
<tr>
<td>Physics</td>
<td>43</td>
<td>45</td>
<td>62</td>
</tr>
</tbody>
</table>

If we drop off the borders, the numerical information contained in the table can be stored in the form of a rectangular array of numbers. This rectangular array is called a matrix. Round or square brackets are used to enclose the numbers which are the elements of the matrix.

In the test score example, the matrix could be represented as:

\[
T = \begin{pmatrix}
45 & 59 & 89 \\
35 & 54 & 83 \\
65 & 65 & 75 \\
43 & 45 & 62 \\
\end{pmatrix}
\]

We shall use the round brackets. Also note that we have designated the matrix by a capital letter which is highlighted.

A matrix has \( r \) rows and \( c \) columns. Matrix \( T \) in our example has 4 rows and 3 columns. We say that this matrix is a ‘4 by 3’ matrix or its order is 4 by 3. When writing the order of a matrix we always write the row number first. You may find it helpful to remember this by thinking of a rectangle.

An element of a matrix is referred to by a double subscript \( ij \) if it occupies a position in the \( i \)-th row and \( j \)-th column. In fact, a general matrix of order \( r \) by \( c \) could be written as:

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1c} \\
a_{21} & a_{22} & \cdots & a_{2c} \\
\vdots & \vdots & \ddots & \vdots \\
a_{r1} & a_{r2} & \cdots & a_{rc}
\end{pmatrix}
\]
Example 1  Let $X = \begin{pmatrix} 8 & -9 \\ 5 & 4 \\ 2 & 1 \end{pmatrix}$. State $x_{21}$

Here $x_{21} = 5$

A matrix may have order 1 by $c$ in which case it is a row matrix or row vector since it only has one row. A matrix with one column is called a column matrix or column vector. This type of matrix has order $r$ by 1. A matrix may also have an equal number of rows and columns, in which case it is referred to as a square matrix. Its order would be $r$ by $r$. A special square matrix is the diagonal matrix in which all elements are zero except those on the principal diagonal, ie the diagonal from top left to bottom right.

Example 2  Let $A = \begin{pmatrix} 4 & -9 \\ 5 & 3 \\ 2 & 1 \end{pmatrix}$, $B = (2 \ -3 \ 0.5)$, $C = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$, $D = \begin{pmatrix} 4 & -1 & 0 \\ 4 & 1 & 0 \\ 3 & 7 & 9 \end{pmatrix}$,

$E = \begin{pmatrix} 2 & 4 & 5 & -1 \\ 4 & 3 & 8 & 3 \end{pmatrix}$, $F = \begin{pmatrix} 5 & 0 \\ 0 & -7 \end{pmatrix}$

State the order of each and identify them as column matrix, row matrix, square matrix, diagonal matrix or rectangular matrix.

$A$ has order 3 by 2. It is a rectangular matrix.

$B$ has order 1 by 3. It is a row matrix.

$C$ has order 2 by 1. It is a column matrix.

$D$ has order 3 by 3. It is a square matrix.

$E$ has order 2 by 4. It is a rectangular matrix.

$F$ has order 2 by 2. It is a diagonal matrix.

Sometimes there is a need to interchange rows and columns, in which case we obtain the transpose of a matrix. If $M$ is the matrix, its transpose is written as $M^T$. Note that when a matrix has order $r$ by $c$, its transpose will have order $c$ by $r$.

Example 3  If $A = \begin{pmatrix} 3 & 4 & -1 \\ 3 & 0 & 8 \end{pmatrix}$, write down the transpose and state the order of each.

$A^T = \begin{pmatrix} 3 \\ 4 \\ -1 \\ 3 \\ 0 \\ 8 \end{pmatrix}$ $A$ has order 2 by 3 while $A^T$ has order 3 by 2.
Activity 7.1

1. State the order of each of the following matrices.

   a) \[
   \begin{pmatrix}
   1 & 2 \\
   3 & 4 
   \end{pmatrix}
   \]

   b) \[
   \begin{pmatrix}
   1 & 0 \\
   0 & 1 
   \end{pmatrix}
   \]

   c) \[
   \begin{pmatrix}
   3 & 3 & 5 \\
   2 & 3 & 5 \\
   4 & -6 & 0 
   \end{pmatrix}
   \]

   d) \[
   \begin{pmatrix}
   1 & -2 & -3 & -4 \\
   6 & 2 & 0 & -1 
   \end{pmatrix}
   \]

2. How many elements are there in a:

   a) \(3 \times 3\) matrix?  b) \(2 \times 2\) matrix?  c) square matrix of order \(n\)?

3. Write down the transpose of each of the matrices in question 1.

4. Write down the 2 by 3 matrix in which \(a_{ij} = i - j\).

7.2 Addition and multiples of matrices

Two matrices are equal if they have the same order and the corresponding elements are equal.

Example 1

If \[
\begin{pmatrix}
  a & b \\
  c & d 
\end{pmatrix}
= \begin{pmatrix}
  -1 & 3 \\
  3 & 4 
\end{pmatrix},
\] what are \(a, b, c\) and \(d\)?

\(a = -1, b = 3, c = 3\) and \(d = 4\).

Matrices can be added provided they have the same order. The sum is obtained by adding corresponding elements.

Example 2

Let \[
A = \begin{pmatrix}
  2 & -1 & 5 \\
  5 & 0 & 2 
\end{pmatrix}
\] and \[
B = \begin{pmatrix}
  5 & 6 & -5 \\
  4 & 0 & -2 
\end{pmatrix}.
\] Obtain \(A + B\).

\[
A + B = \begin{pmatrix}
  2+5 & -1+6 & 5+(-5) \\
  5+4 & 0+0 & 2+(-2) 
\end{pmatrix} = \begin{pmatrix}
  7 & 5 & 0 \\
  9 & 0 & 0 
\end{pmatrix}
\]

Note that the commutative law holds for addition of matrices, ie \(A + B = B + A\).

Also if we let \(O\) stand for the matrix with only zeros, called the zero matrix or null matrix, then \(A + O = O + A = A\).

A matrix can also be multiplied by a constant in which case all elements are multiplied by that constant.
Section 7 Matrices

Example 3  Let \( A = \begin{pmatrix} 2 & -1 & 6 \\ 5 & 0 & 0.5 \end{pmatrix} \). Obtain \( 3A \).

\[
3A = \begin{pmatrix} 3 \times 2 & 3 \times -1 & 3 \times 6 \\ 3 \times 5 & 3 \times 0 & 3 \times 0.5 \end{pmatrix} = \begin{pmatrix} 6 & -3 & 18 \\ 15 & 0 & 1.5 \end{pmatrix}
\]

Of particular interest is the case when the constant is \(-1\) in which case we obtain \((-A)\). This allows us to define matrix subtraction as \( A - B = A + (-B) \).

Example 4  Let \( A = \begin{pmatrix} 2 & -1 & 5 \\ 5 & 0 & 2 \end{pmatrix} \) and \( B = \begin{pmatrix} 5 & 6 & -5 \\ 4 & 0 & -2 \end{pmatrix} \). Obtain \( A - B \).

Here \( A - B = \begin{pmatrix} 2 - 5 & -1 - 6 & 5 - (-5) \\ 5 - 4 & 0 - 0 & 2 - (-2) \end{pmatrix} = \begin{pmatrix} -3 & -7 & 10 \\ 1 & 0 & 4 \end{pmatrix} \)

Activity 7.2

1. Combine the following matrices:
   a) \( \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} -2 & -1 \\ 6 & 0 \end{pmatrix} \)
   b) \( \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} -2 & -1 \\ 6 & 0 \end{pmatrix} \)
   c) \( \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 5 \end{pmatrix} \)

2. Complete: \( \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} \)

3. Solve the equation: \( X - \begin{pmatrix} 1 & 3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 7 & 0 \end{pmatrix} \) where \( X \) is a 2 by 2 matrix.

4. If \( \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix} \) determine \( x \) and \( y \).

5. If \( A = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} -4 & 1 \\ 3 & -2 \end{pmatrix} \) determine:
   a) \( 2A \)
   b) \( 3B \)
   c) \( 2A + 3B \)
   d) \( 2A - 3B \)
7.3 Multiplication of matrices

We start with an example.

**Example 1** There are four teams in a competition and they have all played 12 games with the following results:

<table>
<thead>
<tr>
<th>Team</th>
<th>Win</th>
<th>Draw</th>
<th>Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>6</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>5</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>D</td>
<td>6</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

If a win is worth 3 points, a draw is worth 1 point and a loss 0 points, determine which team is the leader of the competition.

Without knowing anything about matrices we would compute:

- Points Team A: \((6 \times 3) + (3 \times 1) + (3 \times 0) = 21\)
- Points Team B: \((5 \times 3) + (0 \times 1) + (7 \times 0) = 15\)
- Points Team C: \((4 \times 3) + (1 \times 1) + (7 \times 0) = 13\)
- Points Team D: \((6 \times 3) + (2 \times 1) + (4 \times 0) = 20\)

If we represent the information by the three matrices:

\[
M = \begin{pmatrix} 6 & 3 & 3 \\ 5 & 0 & 7 \\ 4 & 1 & 7 \\ 6 & 2 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 21 \\ 15 \\ 13 \\ 20 \end{pmatrix}
\]

then essentially what we have done to get the total for Team A such as element \((1,1)\) of \(T\), is to multiply each 1st row element of \(M\) with the corresponding element in the 1st (and only) column of \(P\).

To get the total for Team B such as element \((2,1)\) of \(T\), we multiply each 2nd row element of \(M\) with the corresponding element in the 1st (and only) column of \(P\). The other team totals are obtained in similar fashion.

This is the way multiplication of matrices is defined. To get element \((i,j)\) of the matrix product \(AB\) the \(i\)-th row elements of matrix \(A\) need to be multiplied in succession by the \(j\)-th column elements of matrix \(B\) and these products are then added.

Matrix multiplication is only defined if the number of columns of the first matrix corresponds to the number of rows in the second. Thus, for example, a 2 by 3 matrix can be multiplied by a 3 by 3 matrix but not vice versa.
Thus matrices can only be multiplied if they line up like in a domino game.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\quad
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
= 
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

The resulting matrix will always have the number of rows of the first matrix and the number of columns of the second. In domino terms, the order of the product is what is left after the middle numbers are cancelled.

Formally:

If \( A \) is a \( r_A \) by \( c_A \) matrix and \( B \) is \( r_B \) by \( c_B \) matrix then if \( c_A = r_B \), \( AB \) is defined and will be a \( r_A \) by \( c_B \) matrix.

Multiplication of matrices requires practice and you will soon learn the benefit of using fingers or a pen to obscure rows and columns not being used.

Let us start with a simple example.

**Example 2** Calculate:

\[
\begin{pmatrix}
3 & 5 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
2 \\
3
\end{pmatrix}
= 
\begin{pmatrix}
3 \cdot 2 + 5 \cdot 3 \\
1 \cdot 2 + 0 \cdot 3
\end{pmatrix}
= 
\begin{pmatrix}
21 \\
2
\end{pmatrix}
\]

**Example 3** If \( X = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix} \), calculate \( X^2 \).

\[
X^2 = 
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 \cdot 1 + 1 \cdot 1 & 1 \cdot 1 + 1 \cdot 0 \\
1 \cdot 1 + 0 \cdot 1 & 1 \cdot 1 + 0 \cdot 0
\end{pmatrix}
= 
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\]

**Example 4** Let \( A = \begin{pmatrix}
1 & -2 & 2 \\
1 & 0 & 2
\end{pmatrix} \) and \( B = \begin{pmatrix}
1 & 0 & 1 \\
3 & -1 & 5 \\
4 & 2 & 0
\end{pmatrix} \).

Determine whether \( AB \) or \( BA \) is defined and compute the product.

\( A \) is 2 by 3 and \( B \) is 3 by 3. Hence \( AB \) is defined but not \( BA \).

\[
AB = 
\begin{pmatrix}
1 \cdot 1 + (-2 \cdot 3) + 2 \cdot 4 & 1 \cdot 0 + (-2 \cdot -1) + 2 \cdot 2 & 1 \cdot 1 + (-2 \cdot 5) + 2 \cdot 0 \\
1 \cdot 1 + 0 \cdot 3 + 2 \cdot 4 & 1 \cdot 0 + 0 \cdot -1 + 2 \cdot 2 & 1 \cdot 1 + 0 \cdot 5 + 2 \cdot 0
\end{pmatrix}
= 
\begin{pmatrix}
3 & 6 & -9 \\
9 & 4 & 1
\end{pmatrix}
\]

It should be obvious that multiplication is generally not commutative, ie \( AB \neq BA \) even if we multiply square matrices. Let us check this.
Example 5  Let \( A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \) and \( B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \). Calculate \( AB \) and \( BA \).

\[
AB = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 + 1 \cdot 1 & 2 \cdot 3 + 1 \cdot 0 \\ 0 \cdot 2 + 3 \cdot 1 & 0 \cdot 3 + 3 \cdot 0 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 3 & 0 \end{pmatrix}
\]

\[
BA = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 + 3 \cdot 0 & 2 \cdot 1 + 3 \cdot 3 \\ 1 \cdot 2 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 4 & 11 \\ 2 & 1 \end{pmatrix}
\]

In practical problems it is sometimes quite difficult to work out which matrices to multiply and in which order this should be done. Sometimes the transpose of a matrix needs to be used.

Consider this example.

Example 6  A builder wants to employ an electrician for the two houses he is currently building. The houses need power points (P) and light fittings (L) according to matrix \( A \). Two electricians \( E_1 \) and \( E_2 \) provide quotes according to matrix \( B \).

\[
A = \begin{pmatrix} P & L \\ H_1 & H_2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} E_1 & E_2 \\ P & L \end{pmatrix} = \begin{pmatrix} 100 & 120 \\ 80 & 70 \end{pmatrix}
\]

Obtain the suitable matrix product that can be used to find the total cost each electrician would charge.

Here we want to be left with a matrix showing Electrician against House. To ‘eliminate’ the P and L components we need to take the transpose of \( B \). It is then a matter of deciding whether to obtain \( AB^T \) or \( B^T A \).

If we want Electricians in rows and House in columns we need \( B^T A \).

\[
B^T A = \begin{pmatrix} 100 & 80 \\ 120 & 70 \end{pmatrix} \begin{pmatrix} 7 & 6 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 100 \cdot 7 + 80 \cdot 4 & 100 \cdot 6 + 80 \cdot 3 \\ 120 \cdot 7 + 70 \cdot 4 & 120 \cdot 6 + 70 \cdot 3 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1020 & 840 \\ 1120 & 930 \end{pmatrix}
\]

Thus Electrician 1 would charge a total of $1860 while Electrician 2 would charge a total of $2050.

Activity 7.3

1. Perform the following multiplications.

   a) \[
   \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}
   \]

   b) \[
   \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
   \]

   c) \[
   \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}
   \]

   d) \[
   \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}
   \]
Section 7 Matrices

2. If \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \), calculate the following:
   a) \( A^2 \)  
   b) \( B^2 \)  
   c) \( 2AB \)  
   d) \( A^2 + B^2 + 2AB \)  
   e) \((A + B)^2\)

3. \( A = \begin{pmatrix} 0 & 5 \\ 3 & -2 \end{pmatrix} \), \( B = \begin{pmatrix} -1 & 8 \\ 0 & -1 \end{pmatrix} \), \( C = \begin{pmatrix} 4 & 1 & -1 \\ 5 & 1 & -2 \end{pmatrix} \), \( D = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -2 & 7 \end{pmatrix} \)

Find all possible products.

4. In a Statistics course three tests are given. Test 1 is worth 20%, Test 2 is worth 50% and Test 3 is worth 30%. A teacher’s markbook has the following information:

<table>
<thead>
<tr>
<th>Student</th>
<th>Test 1 (100%)</th>
<th>Test 2 (100%)</th>
<th>Test 3 (100%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>87</td>
<td>98</td>
<td>65</td>
</tr>
<tr>
<td>Bill</td>
<td>76</td>
<td>43</td>
<td>56</td>
</tr>
<tr>
<td>Jack</td>
<td>54</td>
<td>25</td>
<td>19</td>
</tr>
<tr>
<td>Mary</td>
<td>67</td>
<td>65</td>
<td>43</td>
</tr>
</tbody>
</table>

Use the appropriate matrix multiplication to find, correct to the nearest percent, the final marks for the four students.

5. Consider the matrix \( V \) which gives the vitamin content (in suitable units) of three typical breakfast components:

\[
V = \begin{pmatrix}
A & B_1 & B_2 & C \\
\text{Orange Juice} & 500 & 0.2 & 0 & 129 \\
\text{Oatmeal} & 0 & 0.2 & 0 & 0 \\
\text{Milk} & 1560 & 0.32 & 1.7 & 6
\end{pmatrix}
\]

Also consider the consumption matrix \( C \) (in suitable units) for three days of the week.

\[
C = \begin{pmatrix}
m & t & w \\
\text{Orange Juice} & 2 & 2 & 1 \\
\text{Oatmeal} & 2 & 3 & 2 \\
\text{Milk} & 0.5 & 0.5 & 0.5
\end{pmatrix}
\]

Evaluate the appropriate matrix product and state on what day the greatest amount of Vitamin B1 will be consumed.
7.4 Inverse of a matrix

Earlier it was stated that matrix multiplication is generally not commutative, ie $AB \neq BA$. The exception (apart from the situation when $A = B$) involves the so-called identity matrix, $I$.

The identity matrix is a diagonal matrix with 1’s on the principal diagonals. Thus:

$$
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ etc}
$$

This identity matrix fulfils the role of 1 in ordinary multiplication where, for example,

$$
1 \times 8 = 8 \times 1 = 8
$$

In fact:

$$
AI = IA = A
$$

Let us verify this result.

Example 1  Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Calculate $AI$ and $IA$.

$$
AI = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 1 \times 0 & 2 \times 0 + 1 \times 1 \\ 0 \times 1 + 3 \times 0 & 0 \times 0 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}
$$

$$
IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 0 \times 0 & 1 \times 1 + 0 \times 3 \\ 0 \times 2 + 1 \times 0 & 0 \times 1 + 1 \times 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}
$$

The identity matrix is used extensively in matrix algebra to obtain the multiplicative inverse of a matrix. Let us calculate an inverse matrix.

Example 2  Solve for $a$, $b$, $c$ and $d$ if $\begin{pmatrix} -3 & -7 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

On multiplying we obtain $
\begin{pmatrix} -3a - 7c & -3b - 7d \\ 2a + 5c & 2b + 5d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$
\begin{align*}
-3a - 7c &= 1 \\
2a + 5c &= 0
\end{align*}
$$

and

$$
\begin{align*}
-3b - 7d &= 0 \\
2b + 5d &= 1
\end{align*}
$$

Solving gives $a = (-5)$, $b = (-7)$, $c = 2$ and $d = 3$.

Thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -5 & -7 \\ 2 & 3 \end{pmatrix}$ which is the inverse matrix of $\begin{pmatrix} -3 & -7 \\ 2 & 5 \end{pmatrix}$. 


A matrix dimension analysis will reveal that only square matrices can have inverses.

If we denote the inverse of \( A \) by \( A^{-1} \) then:

\[
AA^{-1} = A^{-1}A = I
\]

Note that an inverse does not always exist. Consider this example.

**Example 3** Let \( A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \). Calculate \( A^{-1} \) if it exists.

We need to obtain \( a, b, c \) and \( d \) so that:

\[
\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

On multiplying we obtain:

\[
\begin{pmatrix} 2a + c & 2b + 1d \\ 4a + 2c & 4b + 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

This leads to:

\[
\begin{align*}
2a + c &= 1 \\
4a + 2c &= 0 \\
2b + d &= 0 \\
4b + 2d &= 1
\end{align*}
\]

Looking at the first pair of equations in detail, we note that \( 4a + 2c = 0 \) equates to \( 2a + c = 0 \). But the first equation in that pair states that \( 2a + c = 1 \). Thus there is no solution and \( A^{-1} \) does not exist.

If a matrix has an inverse it is called **non-singular**. Matrix \( A \) in the previous example is **singular**.

Inverses can be used to solve matrix equations. Solving these equations is very similar to solving algebraic equations. Compare the following:

<table>
<thead>
<tr>
<th>Algebra equation</th>
<th>Matrix equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ax = b )</td>
<td>( AX = B )</td>
</tr>
<tr>
<td>( a^{-1}ax = a^{-1}b )</td>
<td>( A^{-1}AX = A^{-1}B )</td>
</tr>
<tr>
<td>( bx = a^{-1}b )</td>
<td>( IX = A^{-1}B )</td>
</tr>
<tr>
<td>( x = \frac{b}{a} )</td>
<td>( X = A^{-1}B )</td>
</tr>
</tbody>
</table>

In the algebraic equation, \( a^{-1} \) is \( \frac{1}{a} \) but in the matrix equation \( \frac{1}{A} \) has no meaning.

**Example 4** Verify that if \( A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \) then \( A^{-1} = \begin{pmatrix} -0.5 & 0.75 \\ 0.5 & -0.25 \end{pmatrix} \) and solve \( AX = B \)

where \( B = \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix} \)
Now \( AA^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -0.5 & 0.75 \\ 0.5 & -0.25 \end{pmatrix} \)
\[
= \begin{pmatrix} 1 \times -0.5 + 3 \times 0.5 & 1 \times 0.75 + 3 \times -0.25 \\ 2 \times -0.5 + 2 \times 0.5 & 2 \times 0.75 + 2 \times -0.25 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\( X = A^{-1}B = \begin{pmatrix} -0.5 & 0.75 \\ 0.5 & -0.25 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix} \)
\[
= \begin{pmatrix} -0.5 \times 3 + 0.75 \times 4 & -0.5 \times 0 + 0.75 \times 2 \\ 0.5 \times 3 + (-0.25) \times 4 & 0.5 \times 0 + (-0.25) \times 2 \end{pmatrix} = \begin{pmatrix} 1.5 & 1.5 \\ 0.5 & -0.5 \end{pmatrix}
\]

We finish this topic by showing how the inverse of a 2 by 2 matrix can be quickly obtained.

Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). We require \( A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \) so that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

We will find the value of \( w \). The values of \( x, y \) and \( z \) can be found in similar fashion.

Now \( aw + by = 1 \) \( \ldots \ldots \) 1
and \( cw + dy = 0 \) \( \ldots \ldots \) 2

From 2: \( dy = (-cw) \) or \( y = \frac{-cw}{d} \). Substituting in 1: \( aw + b \left( \frac{-cw}{d} \right) = 1 \).

Factorising produces: \( w \left( a - \frac{bc}{d} \right) = 1 \) or \( w \left( \frac{ad - bc}{d} \right) = 1 \).

This gives \( w = \frac{d}{ad - bc} \). Similarly \( x = \frac{-b}{ad - bc} \), \( y = \frac{-c}{ad - bc} \) and \( z = \frac{a}{ad - bc} \).

Thus \( A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \)

The value \( ad - bc \) is called the determinant of \( A \) or \( \det(A) \) while \( \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \) is called the adjoint of \( A \), abbreviated \( \text{adj}(A) \).
Example 5  If \( A = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix} \) obtain \( A^{-1} \).

\[
A^{-1} = \frac{1}{(1\times4 - (-3)\times2)} \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 \\ -0.2 & 0.1 \end{pmatrix}
\]

Remember that a matrix is singular if it does not have an inverse. For a 2 by 2 matrix this will be the case when the determinant \( ad - bc = 0 \).

Example 6  Verify that \( A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \) from Example 3 is singular.

Clearly \( \text{det}(A) = 2 \times 2 - 1 \times 4 = 0 \).

Activity 7.4

1. Show that if \( A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \) then \( A^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \)

2. Find \( a, b, c \) and \( d \) and hence the inverse of \( A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \) if:

\[
\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

3. Determine whether \( \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \) is singular or non singular.

4. If \( A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \) find \( AB^{-1} \) and \( B^{-1}A \).

5. Solve the matrix equation \( AX = B \) where \( A = \begin{pmatrix} 1 & 3 \\ -1 & 4 \end{pmatrix} \) and \( B = \begin{pmatrix} 3 \\ 11 \end{pmatrix} \)

6. Obtain the inverse of \( I \).

7. Use the adjoint-determinant method to obtain the inverse of \( X = \begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix} \)

8. If \( \text{det}(A) = k \), what is \( \text{det}(A^{-1}) \)?
7.5 Linear equations and matrices

Recall that we solved the matrix equation $AX = B$ by pre-multiplying both sides by $A^{-1}$ to obtain:

$$A^{-1}AX = A^{-1}B$$

from which $IX = A^{-1}B$

and hence $X = A^{-1}B$.

A set of linear equations can be put into the form of a matrix equation which will then enable us to solve the set simultaneously. Let us illustrate the process with an example.

**Example 1** Solve the following set of simultaneous linear equations:

$$2x + 3y = 8$$
$$3x + 5y = 13$$

The equations can be written in matrix form thus:

$$\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

because on multiplying the matrices on the left-hand side gives:

$$\begin{pmatrix} 2x + 3y \\ 3x + 5y \end{pmatrix}.$$ 

Now the inverse of $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ is $\frac{1}{10-9} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$.

Pre-multiplying both sides of the matrix equation by this inverse gives:

$$\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \times 8 + (-3) \times 13 \\ (-3) \times 8 + 2 \times 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Then $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and hence $x = 1$ and $y = 2$. 
Example 2  Solve the following set of simultaneous linear equations:
\[
\begin{align*}
2x + y + 2z &= 3 \\
4x + 3y + 3z &= 8 \\
2x + 2y - z &= 7
\end{align*}
\]
given that the inverse of
\[
\begin{pmatrix}
2 & 1 & 2 \\
4 & 3 & 3 \\
2 & 2 & -1
\end{pmatrix}
\]
is
\[
\begin{pmatrix}
9 & -5 & 3 \\
-10 & 6 & -2 \\
-2 & 2 & -2
\end{pmatrix}
\]

In matrix form the three equations can be written as:
\[
\begin{pmatrix}
2 & 1 & 2 \\
4 & 3 & 3 \\
2 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
3 \\
8 \\
7
\end{pmatrix}
\]

Then
\[
\begin{pmatrix}
9 & -5 & 3 \\
-10 & 6 & -2 \\
-2 & 2 & -2
\end{pmatrix}^{-1}
\begin{pmatrix}
2 & 1 & 2 \\
4 & 3 & 3 \\
2 & 2 & -1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
9 & -5 & 3 \\
-10 & 6 & -2 \\
-2 & 2 & -2
\end{pmatrix}
^{-1}
\begin{pmatrix}
3 \\
8 \\
7
\end{pmatrix}
\]

or
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
9 & -5 & 3 \\
-10 & 6 & -2 \\
-2 & 2 & -2
\end{pmatrix}
^{-1}
\begin{pmatrix}
3 \\
8 \\
7
\end{pmatrix}
\]

or
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
27 - 40 + 21 \\
-30 + 48 - 14 \\
-6 + 16 - 14
\end{pmatrix}
\]

or
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
8 \\
4 \\
-4
\end{pmatrix}
\]

From which we derive that \(x = 2\), \(y = 1\) and \(z = (-1)\).

Activity 7.5

1. Express the simultaneous equations \(2x - 3y = 9\) and \(5x - 7y = 22\) in matrix form. Solve the above equations using matrices, given that the inverse of the matrix:
\[
\begin{pmatrix}
2 & -3 \\
5 & -7
\end{pmatrix}
\]
is
\[
\begin{pmatrix}
-7 & 3 \\
-5 & 2
\end{pmatrix}
\]

2. Solve the simultaneous equations \(5x - 2y = 6\) and \(3x - y = 5\) by the inverse matrix method.
3. The matrix \( \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} \) has inverse \( \mathbf{A}^{-1} = \frac{1}{2} \begin{pmatrix} 7 & -6 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{pmatrix} \).

Use this information to find the values of \( x, y \) and \( z \) which satisfy:
\[
\begin{align*}
x + 2y + 3z &= 2 \\
x + 3y + 4z &= 2 \\
x + 4y + 3z &= 4
\end{align*}
\]

4. If \( \mathbf{M} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{pmatrix} \) and \( \mathbf{N} = \frac{1}{2} \begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix} \)

a) Find the matrix \( \mathbf{MN} \).

b) Write down the matrix \( \mathbf{M}^{-1} \).

c) Use the results to solve the following system:
\[
\begin{align*}
x + 2y - z &= -1 \\
3x + 5y - z &= 2 \\
-2x - y - 2z &= 9
\end{align*}
\]

## 7.6 Determinants

Associated with every square matrix there is a ‘determined’ value called the **determinant**. We already encountered the determinant for a second order square matrix \( \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and defined it as \( \text{det}(\mathbf{A}) = ad - bc \).

For a more general definition we may let a matrix be \( \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \) in which case the determinant is \( a_1b_2 - a_2b_1 \).

We shall mainly use the ‘det’ abbreviation but sometimes we have use for the notation \( \Delta \) (delta). We can also write a determinant with straight vertical brackets \( | | \).

Thus:
\[
\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1
\]
Example 1 Calculate: \[
\begin{bmatrix}
1 & -3 \\
4 & -1
\end{bmatrix}
\]
\[
\begin{align*}
\begin{vmatrix}
1 & -3 \\
4 & -1
\end{vmatrix} &= (1)(-1) - (-3)(4) = -1 + 12 = 11
\end{align*}
\]

A third order determinant is an extension of this cross multiplication formation.

In fact

\[
\begin{vmatrix}
a_{1} & b_{1} & c_{1} \\
a_{2} & b_{2} & c_{2} \\
a_{3} & b_{3} & c_{3}
\end{vmatrix} = a_{1} \begin{vmatrix}
b_{2} & c_{2} \\
b_{3} & c_{3}
\end{vmatrix} - a_{2} \begin{vmatrix}
b_{1} & c_{1} \\
b_{3} & c_{3}
\end{vmatrix} + a_{3} \begin{vmatrix}
b_{1} & c_{1} \\
b_{2} & c_{2}
\end{vmatrix}
\]

Example 2 Calculate: \[
\begin{bmatrix}
2 & 4 & -1 \\
7 & 1 & 6 \\
5 & 3 & 0
\end{bmatrix}
\]
\[
\begin{align*}
\begin{vmatrix}
2 & 4 & -1 \\
7 & 1 & 6 \\
5 & 3 & 0
\end{vmatrix} &= 2 \begin{vmatrix}
1 & 6 \\
3 & 0
\end{vmatrix} - 7 \begin{vmatrix}
4 & -1 \\
3 & 0
\end{vmatrix} + 5 \begin{vmatrix}
4 & -1 \\
1 & 6
\end{vmatrix} \\
&= 2(-18) - 7(3) + 5(25) = 68
\end{align*}
\]

By expanding \[
\begin{vmatrix}
b_{2} & c_{2} \\
b_{3} & c_{3}
\end{vmatrix}, \begin{vmatrix}
b_{1} & c_{1} \\
b_{3} & c_{3}
\end{vmatrix}, \text{ and } \begin{vmatrix}
b_{1} & c_{1} \\
b_{2} & c_{2}
\end{vmatrix}
\]
the so-called minors of \(a_{1}, a_{2}\) and \(a_{3}\), we can derive that:

\[
\begin{vmatrix}
a_{1} & b_{1} & c_{1} \\
a_{2} & b_{2} & c_{2} \\
a_{3} & b_{3} & c_{3}
\end{vmatrix} = a_{1}b_{2}c_{3} - a_{1}b_{3}c_{2} - a_{2}b_{1}c_{3} + a_{2}b_{3}c_{1} + a_{3}b_{1}c_{2} - a_{3}b_{2}c_{1}
\]

It is hard to remember this expression but there is a ‘trick’ called Sarrus’ Rule. This rule, due to the French Mathematician J.P. Sarrus, uses the following steps.

- Rewrite the first two columns to the right to create a 3 by 5 matrix.
- 3 distinct diagonals from top left to bottom right are created. The numbers in each diagonal are multiplied and their products are added to produce Diagonal Sum 1.
- 3 distinct diagonals from bottom left to top right are created. The numbers in each diagonal are multiplied and their products are added to produce Diagonal Sum 2.
- The determinant is then equal to Diagonal Sum 1 – Diagonal Sum 2.
Schematically the rule works as follows:

For Diagonal Sum 1:

For Diagonal Sum 2:

Note: The diagonals move from left to right. Sometimes the rule is remembered by letting the sum of the products be positive in a downward direction and negative in an upward direction.

Example 3 Calculate:

\[
\begin{vmatrix}
2 & 4 & -1 \\
7 & 1 & 6 \\
5 & 3 & 0
\end{vmatrix}
\]

using Sarrus’ Rule.

Rewriting the first two columns:

\[
\begin{vmatrix}
2 & 4 & -1 \\
7 & 1 & 6 \\
5 & 3 & 0
\end{vmatrix}
\]

Diagonal Sum 1 = (2)(1)(0) + (4)(6)(5) + (−1)(7)(3) = 99

Diagonal Sum 2 = (5)(1)(−1) + (3)(6)(2) + (0)(7)(4) = 31

Then \[
\begin{vmatrix}
2 & 4 & -1 \\
7 & 1 & 6 \\
5 & 3 & 0
\end{vmatrix}
\] = 99 − 31 = 68.

Activity 7.6

1. Evaluate the following determinants:
   a) \[
   \begin{vmatrix}
   5 & 2 \\
   3 & 6
   \end{vmatrix}
   \]
   b) \[
   \begin{vmatrix}
   7 & 4 \\
   5 & 2
   \end{vmatrix}
   \]
   c) \[
   \begin{vmatrix}
   6 & 8 \\
   2 & 5
   \end{vmatrix}
   \]

2. Evaluate the following determinants:
   a) \[
   \begin{vmatrix}
   2 & 4 & 1 \\
   4 & 5 & 3 \\
   8 & 7 & 2
   \end{vmatrix}
   \]
   b) \[
   \begin{vmatrix}
   2 & 4 & 3 \\
   9 & 6 & 2 \\
   2 & 7 & 5
   \end{vmatrix}
   \]
   c) \[
   \begin{vmatrix}
   8 & 4 & 2 \\
   9 & 6 & 7 \\
   1 & 3 & 5
   \end{vmatrix}
   \]
3. Consider \( A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \\ 2 & 2 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix} \).

Confirm that \( B^{-1} = \begin{pmatrix} -0.5 & -2.5 & 1.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & 3.5 & -1.5 \end{pmatrix} \) and find:

a) \( \det A \)  

b) \( \det B^{-1} \)  

c) \( \det (AB^{-1}) \)  

d) \( (\det A) \times (\det B^{-1}) \)

What do these results suggest?

### 7.7 Inverse of a matrix using co-factors

In this topic we describe how the inverse of a high order matrix such as a matrix of order 3 by 3 may be found.

There are various methods for obtaining the inverse of a high order matrix. We shall look at only one method which uses so-called co-factors. The method is quite involved but is detailed here because it is essentially the method we already encountered to find the inverse of an order 2 by 2 matrix.

Let us first define what a co-factor of a matrix element is. This is like a minor which we encountered earlier. The minor of an element is the determinant of the matrix which results when all elements in the same row and in the same column of that element are deleted.

Consider \( A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \).

The minor of \( a_1 \), denoted \( |A_1| \) = \( \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \)

The minor of \( b_3 \), denoted \( |B_3| \) = \( \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \)

The co-factor of an element is the minor of that element together with a positive or negative sign. The sign is determined according to the following rule.

- The sign is positive if the row plus the column number of the element is even.
- The sign is negative if the row plus the column number of the element is odd.

For an order 3 by 3 matrix, this pattern is \( \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \).
The corresponding matrix of co-factors is:
\[
\begin{pmatrix}
+A_1 & -B_1 & +C_1 \\
-A_2 & +B_2 & -C_2 \\
+A_3 & -B_3 & +C_3
\end{pmatrix}
\]

Finally we define the transpose of this matrix as the **adjoint**, abbreviated adj. Thus:
\[
\text{adj}(A) = \begin{pmatrix}
+A_1 & -A_2 & +A_3 \\
-B_1 & +B_2 & -B_3 \\
+C_1 & -C_2 & +C_3
\end{pmatrix}
\]

If we multiply this matrix by \( A \) we obtain an interesting result:
- **Consider element (1, 1):**
  \[
  \begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
  \end{pmatrix}
  \begin{pmatrix}
  +A_1 & -A_2 & +A_3 \\
  -B_1 & +B_2 & -B_3 \\
  +C_1 & -C_2 & +C_3
  \end{pmatrix}
  = \begin{pmatrix}
  a_1 + b_1 A_1 - c_1 C_1 & \cdots \\
  a_1 + b_1 A_2 - c_1 C_2 & \cdots \\
  a_1 + b_1 A_3 - c_1 C_3 & \cdots
  \end{pmatrix}
  \]
  Now:
  \[
  a_1 \times |A_1| - b_1 \times |B_1| + c_1 \times |C_1| = a_1 b_2 c_3 - a_2 b_3 c_1 + a_3 b_1 c_2 - b_1 a_2 c_1 + b_2 a_3 c_2 + b_3 a_1 c_3
  \]
  \[
  = A
  \]

- **Consider element (1, 2):**
  \[
  \begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
  \end{pmatrix}
  \begin{pmatrix}
  +A_1 & -A_2 & +A_3 \\
  -B_1 & +B_2 & -B_3 \\
  +C_1 & -C_2 & +C_3
  \end{pmatrix}
  = \begin{pmatrix}
  \cdots & a_1 - b_1 A_1 + c_1 C_1 & \cdots \\
  \cdots & a_1 - b_1 A_2 + c_1 C_2 & \cdots \\
  \cdots & a_1 - b_1 A_3 + c_1 C_3 & \cdots
  \end{pmatrix}
  \]
  Now:
  \[
  a_1 \times |A_2| - b_1 \times |B_2| + c_1 \times |C_2| = -a_1 b_2 c_1 + a_3 b_1 c_3 - b_1 a_3 c_3 + a_2 b_1 c_2 - b_2 a_2 c_2 + a_3 b_2 c_2
  \]
  \[
  = 0
  \]

Continuing with all elements we obtain:
\[
\begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3
  \end{pmatrix}
  \begin{pmatrix}
  +A_1 & -A_2 & +A_3 \\
  -B_1 & +B_2 & -B_3 \\
  +C_1 & -C_2 & +C_3
  \end{pmatrix}
  = \begin{pmatrix}
  |A| & 0 & 0 \\
  0 & |A| & 0 \\
  0 & 0 & |A|
  \end{pmatrix}
  = |A|
  \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{pmatrix}
  = |A| I
  \]
Section 7 Matrices

Symbolically: \[ A \times \text{adj}(A) = |A| I \]

or \[ A \times \frac{\text{adj}(A)}{|A|} = I \]

and hence:

\[ A^{-1} = \frac{\text{adj}(A)}{|A|} \]

We gave this result earlier for an order 2 by 2 matrix but it has now meaning for any square matrix.

Example 1 Derive the adjoint for the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

The matrix of minors is \( \begin{pmatrix} d & c \\ b & a \end{pmatrix} \).

The matrix of co-factors is \( \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \).

The adjoint (the transpose of the co-factor matrix) is \( \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

Example 2 Calculate the inverse of \( A = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \).

\[ |A| = (2)(-2) - (-1)(-3) + (0)(0) = (-7) \]

The matrix of co-factors is \( \begin{pmatrix} -2 & -(-1) & -1 \\ -(-3) & 2 & -2 \\ 0 & -(7) & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 \\ 3 & 2 & -2 \\ 0 & -7 & 0 \end{pmatrix} \).

Consequently, \( \text{adj}(A) = \begin{pmatrix} -2 & 3 & 0 \\ 1 & 2 & -7 \\ -1 & -2 & 0 \end{pmatrix} \).

Then \( A^{-1} = \frac{1}{-7} \begin{pmatrix} -2 & 3 & 0 \\ 1 & 2 & -7 \\ -1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{-3}{7} & 0 \\ -\frac{1}{7} & \frac{2}{7} & 1 \\ \frac{1}{7} & \frac{2}{7} & 0 \end{pmatrix} \).
Example 3

Solve the system:

\[
\begin{align*}
y + 2z &= (-5) \\
5x + y - 3z &= 15 \\
3x + y - 2z &= 5
\end{align*}
\]

Let \( A = \begin{pmatrix} 0 & 1 & 2 \\ 5 & 1 & -3 \\ 3 & 1 & -2 \end{pmatrix} \), \( X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) and \( B = \begin{pmatrix} -5 \\ 15 \\ 5 \end{pmatrix} \)

\[ |A| = (0)(1) - (5)(-4) + (3)(-5) = 5 \]

The matrix of co-factors is

\[
\begin{pmatrix}
1 & -(-1) & 2 \\
-(-4) & -6 & -(-3) \\
-5 & -(-10) & -5
\end{pmatrix}
\]

Consequently, \( \text{adj}(A) = \begin{pmatrix} 1 & 4 & -5 \\ 1 & -6 & 10 \\ 2 & 3 & -5 \end{pmatrix} \).

Then \( A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 4 & -5 \\ 1 & -6 & 10 \\ 2 & 3 & -5 \end{pmatrix} \begin{pmatrix} 0.2 & 0.8 & -1 \\ 0.2 & -1.2 & 2 \\ 0.4 & 0.6 & -1 \end{pmatrix} \).

Now if \( AX = B \) then \( A^{-1}AX = A^{-1}B \)

from which \( IX = A^{-1}B \)

and hence \( X = A^{-1}B \).

Thus \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.2 & 0.8 & -1 \\ 0.2 & -1.2 & 2 \\ 0.4 & 0.6 & -1 \end{pmatrix} \begin{pmatrix} -5 \\ 15 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \\ 2 \end{pmatrix} \).

The solution therefore is \( x = 6, y = (-9) \) and \( z = 2 \).

As you will agree, this is not a particularly quick way to solve three equations in three unknowns and this method is therefore generally not used. Cramer's rule which we shall discuss next is more 'user friendly'. However, to derive it we need the concept of co-factors.
**Activity 7.7**

1. Obtain the inverse of this matrix \[
\begin{pmatrix}
2 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{pmatrix}
\] using the method of co-factors.

2. Consider \[
A = \begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 2 \\
-2 & 1 & -1
\end{pmatrix}
\].

Find:
   a) \[|A|\]
   b) \[A^{-1}\]
   c) \[|A|^2\]
   d) \[|A^{-1}|\]
   e) \[\text{adj}(A)\]
   f) \[\frac{1}{|A|}\]

What conclusion do you reach about c) and e) and d) and f)?

3. Solve the following systems of linear equations using the inverse matrix method.
   a) \[
   \begin{align*}
   x + 2y + 3z &= 2 \\
   x + 3y + 4z &= 2 \\
   x + 4y + 3z &= 4
   \end{align*}
   \]
   b) \[
   \begin{align*}
   3x - y + 3z &= 5 \\
   x + 2y + 4z &= -2 \\
   3x - y + 4z &= 4
   \end{align*}
   \]

---

**7.8 Cramer’s rule**

As we found, solving a system of three linear equations in three unknowns using the inverse matrix method is a time-consuming process. For this reason, we shall explore another method called Cramer’s Rule.

The rule is named after the Swiss Mathematician Gabriel Cramer (1704–1752) but was discovered by Leibnitz, the co-inventor of calculus. The rule can be shown to be correct using a method involving co-factors.

We shall illustrate that for a 2 by 2 system of linear equations but it can be applied to any \(n\) by \(n\) system.

Consider the following set of linear equations:
\[
\begin{align*}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{align*}
\]

Then as before:
\[
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
= 
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
\]

If we let \[
A = \begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix},
X = \begin{pmatrix}
x \\
y
\end{pmatrix}
\text{and } C = \begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
\text{ then as shown before } X = A^{-1}C.
\]

When we use \[
A^{-1} = \frac{\text{adj}(A)}{|A|}
\] we obtain \[
X = \frac{\text{adj}(A)}{|A|} C
\]
Now \( |A| = a_1b_2 - a_2b_1 \) and \( \text{adj}(A) = \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \)

Hence \( X = \frac{1}{a_1b_2 - a_2b_1} \begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{a_1b_2 - a_2b_1} \begin{pmatrix} b_2c_1 - b_1c_2 \\ -a_2c_1 + a_1c_2 \end{pmatrix} \)

This leads to \( \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \\ \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \end{pmatrix} \) or \( x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \) and \( y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} \)

Or, in determinant form: \( x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \) and \( y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \)

If we let \( \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \) and \( \Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \)

Then: \( x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \)

Notice that to obtain \( \Delta_x \) we must replace the first column in \( \Delta \) by \( c_1 \) and \( c_2 \).
Thus the ‘x coefficients’ must be replaced by the ‘answer coefficients’.
In a similar way we can obtain \( \Delta_y \). This time we must replace the ‘y coefficients’ with the ‘answer coefficients’.

**Example 1**

Solve \( \begin{cases} 2x + 3y = 8 \\ 3x + 5y = 13 \end{cases} \)

\( \Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \)

\( \Delta_x = \begin{vmatrix} 8 & 3 \\ 13 & 5 \end{vmatrix} = 40 - 39 = 1 \)

\( \Delta_y = \begin{vmatrix} 2 & 8 \\ 3 & 13 \end{vmatrix} = 26 - 24 = 2 \)

Hence \( x = \frac{1}{1} = 1 \) and \( y = \frac{2}{1} = 2 \)
Cramer’s Rule extended to a system of three linear equations in three unknowns becomes:

\[
x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}
\]

where

\[
\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_x = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}
\]

Example 2

Solve the following system using Cramer’s Rule.

\[
\begin{align*}
x + 2y - z &= -1 \\
3x + 5y - z &= 2 \\
-2x - y - 2z &= 9
\end{align*}
\]

\[
\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{vmatrix} = -2, \quad \Delta_x = \begin{vmatrix} 1 & -1 & -1 \\ 3 & 2 & -1 \\ -2 & 9 & -2 \end{vmatrix} = -34, \quad \Delta_y = \begin{vmatrix} -1 & 2 & -1 \\ 2 & 5 & -1 \\ 9 & -1 & -2 \end{vmatrix} = 48, \quad \Delta_z = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ -2 & 1 & 9 \end{vmatrix} = -22
\]

Hence \(x = \frac{48}{-2} = -24\), \(y = \frac{-34}{-2} = 17\), and \(z = \frac{-22}{-2} = 11\).

Activity 7.8

1. Use Cramer’s rule to solve:
   a) \(3x + 2y = 8\) \quad \text{b) } 3x - 2y = 12
   
   4x - y = 5

2. Solve the following systems of linear equations using Cramer’s Rule.
   a) \(x + 2y - z = 3\) \quad \text{b) } 5x + 3y + 7z = 0
   
   2x + 6y + 5z = (-2) \quad 8x + 7y + 10z = 1

3. A boat takes 4 hours to make a trip downriver, with the current, and 6 hours to make the return trip upriver, against the current. The round trip is 48 km. Find the speed of the current and the speed of the boat in still water.

4. A jeweller makes two rings, one containing 8 g of gold and 4 g of silver, the other containing 6 g of gold and 6 g of silver. Materials for the first ring cost $104, and for the second cost $84. Find the price per gram of gold and silver.
5. Mary bought a set of beauty products which consists of a cleanser, a toner and a moisturiser. Each mL of cleanser contains 2 units of collagen, 3 units of elastin and 1 unit of witchhazel. Each mL of toner contains 1 unit of collagen, 1 unit of elastin and 5 units of witchhazel. Each mL of moisturiser contains 4 units of collagen, 3 units of elastin and 1 unit of witchhazel.

How many mL of each product should Mary use on her face each night if she needs exactly 34 units of collagen, 31 units of elastin and 57 units of witchhazel?

6. The graph of the parabola \( y = ax^2 + bx + c \) passes through the points (2, 0), (0, 0) and (-1, 3). Find the equation.

7. In an electrical circuit, using Kirchoff’s laws, the following equations may be derived:

\[
R_1 i_1 + R_2 i_1 + R_3 i_3 + \varepsilon_1 = 0, \quad R_3 i_3 + R_4 i_2 + \varepsilon_2 = 0 \quad \text{and} \quad i_1 + i_2 = i_3
\]

If \( R_1 = 2 \, \Omega \), \( R_2 = 3 \, \Omega \), \( R_3 = 5 \, \Omega \), \( R_4 = 3 \, \Omega \), \( \varepsilon_1 = 15 \, V \) and \( \varepsilon_2 = 2 \, V \), find the currents \( i_1 \), \( i_2 \) and \( i_3 \) (in amperes) in each branch of the circuit.

8. Find A, B and C such that

\[
\frac{5x + 7}{x^3 + 2x^2 - x - 2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{x-1}
\]

Note: \( x^3 + 2x^2 - x - 2 \) factors into \((x + 2)(x + 1)(x - 1)\). This process is called the partial fraction decomposition of \( \frac{5x + 7}{x^3 + 2x^2 - x - 2} \), an important process in Calculus.
7.9 Some applications of matrices

There are many applications of matrices. Without going into any great detail, we will look at three.

1. Digraphs

Many types of diagrams such as electrical circuits, transport routes and family trees can be represented by figures that are made up of points and arcs. In mathematics, such a diagram is called a graph or, more properly, a digraph from directed graph.

It is possible to represent such a graph by means of a matrix, called the adjacency matrix. Look at the diagrams below and their associated adjacency matrices.

1. A
   B
   C

   \[
   \begin{pmatrix}
   A & 0 & 1 \\
   B & 1 & 0 \\
   C & 1 & 1 \\
   \end{pmatrix}
   \]

2. P Q R S

   \[
   \begin{pmatrix}
   P & 0 & 2 & 0 & 0 \\
   Q & 2 & 0 & 0 & 0 \\
   R & 0 & 0 & 0 & 1 \\
   S & 0 & 0 & 1 & 1 \\
   \end{pmatrix}
   \]

3. X Y

   \[
   \begin{pmatrix}
   X & 1 & 3 \\
   Y & 3 & 1 \\
   \end{pmatrix}
   \]

4. A B C

   \[
   \begin{pmatrix}
   A & 0 & 1 \\
   B & 1 & 0 \\
   C & 0 & 2 \\
   \end{pmatrix}
   \]

Notice that each graph is a square matrix. The first three matrices are symmetric with respect to the principal diagonal. Graph 1 is an example of a complete graph because each point (sometimes called vertex or node) is connected to an adjacent point by a path (sometimes called arc or edge). The second graph drawn is called a disconnected graph.
If there are two paths from one point to another, then a ‘2’ is placed in the corresponding entry. There can also be loops such as in the third graph; however, strictly speaking, a graph with a loop is not a digraph.

If there is a restriction in a particular direction, arrows are used such as in the fourth graph.

**Example 1** Consider Graph 4 and its associated adjacency matrix \( X \):

\[
X = \begin{pmatrix}
A & B & C \\
A & 0 & 1 & 0 \\
B & 1 & 0 & 1 \\
C & 0 & 2 & 0
\end{pmatrix}
\]

If \( X \) represents the ‘one-step’ paths, compute \( X^2 \) the matrix of two-step paths and explain why entry (1,1) now equals 1.

\[
X^2 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
2 & 0 & 0
\end{pmatrix}
\]

Entry (1,1) is 1 because there is a two-step path from A to A, i.e., \( A \rightarrow B \) and then \( B \rightarrow A \).

### 2. Cryptography

Cryptography involves encrypting and decrypting data so that a third party cannot intercept and read the data.

The **encryption process** involves the following steps.

- Convert the text of the message into a stream of numerical values.
- Place the data into a matrix.
- Multiply the data by the encoding matrix.
- Convert the matrix into a stream of numerical values that contains the encrypted message.

The **decryption process** is more complicated and involves these steps.

- Place the encrypted stream of numbers that represents an encrypted message into a matrix.
- Multiply by the decoding matrix. The decoding matrix is the inverse of the encoding matrix.
- Convert the matrix into a stream of numbers.
- Convert the numbers into the text of the original message.
Example 2  Using the numeric scheme Space = 0, A = 1, B = 2, ..., Y = 25, and Z = 26
and the encryption matrix (encoding matrix) \( E = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} \):

(a) Encrypt the message ‘Buy Now’.

(b) Decrypt 41, 17, 80, (−40), 41, 17, 92, (−46).

(a) The message ‘Buy Now’ becomes 2, 22, 25, 0, 14, 15, 23.

Since there are seven pieces of data and the encryption matrix is of order 2 by 2, we use a matrix of order 4 by 2 and fill the last spot with a space to make the matrix complete.

Let’s call the original, unencrypted data matrix \( A \).

Then \( A = \begin{pmatrix} 2 & 22 \\ 25 & 0 \\ 14 & 15 \\ 23 & 0 \end{pmatrix} \)

Now \( E = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} \) and

\[
AE = \begin{pmatrix} 2 & 22 \\ 25 & 0 \\ 14 & 15 \\ 23 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -14 & 62 \\ 100 & -50 \\ 41 & 17 \\ 92 & -46 \end{pmatrix}
\]

The message has become (−14), 62, 100, (−50), 41, 17, 92, (−46).

(b) The code 41, 17, 80, (−40), 41, 17, 92, (−46) in matrix form is:

\[
AE = \begin{pmatrix} 41 & 17 \\ 80 & -40 \\ 41 & 17 \\ 92 & -46 \end{pmatrix}
\]

To decrypt the message we need the inverse of the encryption matrix \( E \).

This is \( E^{-1} = \frac{1}{12 - 2} \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{pmatrix} \)

Now \( AEE^{-1} = \begin{pmatrix} 41 & 17 \\ 80 & -40 \\ 41 & 17 \\ 92 & -46 \end{pmatrix} \begin{pmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{pmatrix} = \begin{pmatrix} 14 & 15 \\ 20 & 0 \\ 14 & 15 \\ 23 & 0 \end{pmatrix} \)

The message was 14, 15, 20, 0, 14, 15, 23, 0 or ‘Not Now’.
3. Linear transformation

Matrices can be used to change the coordinates of a point \( P(x, y) \) to a point \( P'(x', y') \). We say that \( P' \) is the image of \( P \) under the transformation and we write \( (x, y) \to (x', y') \).

**Example 3**  Consider matrix transformation matrix \( M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

Use this matrix to transform the trapezium with coordinates \( A(2, 1), B(0, 1), C(1, 0) \) and \( D(2, 0) \) and interpret your result.

![Diagram of a trapezium with coordinates labeled A, B, C, D.](image)

Applying the transformation: 
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 & 2 \\ -1 & -1 & 0 & 0 \end{pmatrix}
\]

The points have become \( A'(2, -1), B'(0, -1), C'(1, 0) \) and \( D'(2, 0) \).

![Diagram of a reflected trapezium with coordinates A', B', C', D'.](image)

The trapezium has been reflected in the \( x \) axis.
Activity 7.9

1. Write down the adjacency matrix $M$ for this graph.

```
A
B
C
D
```

What information does $M^2$ give about this graph?

2. Given the adjacency matrix $A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{pmatrix}$.

a) Find $A^2$, $A^3$ and $A + A^2 + A^3$.

b) If $A + A^2 + A^3$ does not have any zero entries, the graph represented by $A$ is said to be connected.

Is the graph represented by $A$ connected?

3. Using the numeric scheme Space = 0, A = 1, B = 2, ..., Y = 25, and Z = 26 and the encryption matrix (encoding matrix) $E = \begin{pmatrix}
8 & 5 \\
12 & 16
\end{pmatrix}$:

a) Encrypt the message 'They Know'.

b) Decrypt 192 222 100 105.

4. Consider matrix transformation matrix $M = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}$.

Use this matrix to transform the trapezium with coordinates $A(2, 1)$, $B(0, 1)$, $C(1, 0)$ and $D(2, 0)$ and interpret your result.

5. A matrix transformation maps the point $(1, 0)$ to the point $(0, -1)$ and the point $(0, 1)$ to the point $(1, 0)$:

a) What is the matrix of this transformation?

b) What effect does this transformation have on the trapezium ABCD of the previous question?
7.10 Some applications of determinants

1. Equation of a circle

Given three points, the equation of a circle may be derived using determinants.

Using determinants the coefficients $D$, $E$ and $F$ in the general equation of the circle:

$$x^2 + y^2 + Dx + Ey + F = 0$$

can be calculated from:

$$D = \begin{vmatrix} p & y_1 & 1 \\ q & y_2 & 1 \\ r & y_3 & 1 \end{vmatrix} \quad E = \begin{vmatrix} x_1 & p & 1 \\ x_2 & q & 1 \\ x_3 & r & 1 \end{vmatrix} \quad F = \begin{vmatrix} x_1 & y_1 & p \\ x_2 & y_2 & q \\ x_3 & y_3 & r \end{vmatrix}$$

where: $K = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$p = -(x_1^2 + y_1^2) \quad q = -(x_2^2 + y_2^2) \quad r = -(x_3^2 + y_3^2)$

It can be shown that $K$ is double the area of the triangle formed by the three points. Thus:

$$\text{Area of triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where this value is always taken as positive.

Example 1 Find the equation of the circle through the points $(1, 2)$, $(-2, 0)$ and $(-1, -5)$ and calculate the area formed of the triangle formed by the three points.

Here:

$$K = \begin{vmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \\ -1 & -5 & 1 \end{vmatrix} = 17$$

$p = -((1)^2 + (2)^2) = (-5)$

$q = -((-2)^2 + 0^2) = (-4)$

$r = -((-1)^2 + (-5)^2) = (-26)$

Then: $D = \begin{vmatrix} -5 & 2 & 1 \\ -4 & 0 & 1 \\ -26 & -5 & 1 \end{vmatrix} = \begin{vmatrix} -49 \\ 17 \end{vmatrix}$
2. Equation of a plane

Given three points, the equation of a plane through them may be derived using determinants.

Using determinants, the coefficients $A$, $B$, $C$ and $D$ in the general equation of the plane

$$Ax + By + Cz + D = 0$$

can be calculated from:

$$A = \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$$

$$B = -\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$D = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Example 2  Find the equation of the plane through $(2, 1, 3)$, $(-1, -2, 4)$ and $(4, 2, 1)$.

Here

$$A = \begin{vmatrix} 1 & 3 & 1 \\ -2 & 4 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 5$$

$$B = -\begin{vmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \\ 4 & 1 & 1 \end{vmatrix} = (-4)$$

$$C = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -2 & 1 \\ 4 & 2 & 1 \end{vmatrix} = 3$$

$$D = -\begin{vmatrix} 2 & 1 & 3 \\ -1 & -2 & 4 \\ 4 & 2 & 1 \end{vmatrix} = (-15)$$

Hence the equation of the plane is $5x - 4y + 3z - 15 = 0$. 

Hence the equation of the circle is:

$$x^2 + y^2 - \frac{49}{17}x + \frac{65}{17}y - \frac{166}{17} = 0$$

or $17x^2 + 17y^2 - 49x + 65y - 166 = 0$

Since area = \(\frac{1}{2}K\), the area formed equals \(\frac{1}{2} \times 17 = 8.5\).
Activity 7.10

1. Verify that the equation of the circle through (5,3), (6, 2) and (3, 4) is
   \[ x^2 + y^2 - 3x + 3y - 28 = 0. \]
   Also show that the area of the triangle formed by the three points is 0.5.

2. Using determinants, find the in-centre of the triangle with vertices
   \( A(0, -3) \), \( B(12, 15) \) and \( (-7.5, 2) \).

3. Calculate the area of the triangle formed by the vertices with coordinates
   \( A(3, -2) \), \( B(-4, 1) \) and \( C(8, 3) \).

4. Find the equation of the plane through \( (1, 1, -1) \), \( (-2, -2, 2) \) and \( (1, -1, 2) \).

5. A plane passes through \( (2, 0, 0) \), \( (0, -4, 0) \) and \( (8, 0, 0) \). Find its equation.

6. Under the transformation with matrix \[
   \begin{pmatrix}
   2 & 1 \\
   1 & 3
   \end{pmatrix}
\]
   the unit square OABC
   [ie the square with vertices O(0, 0), A(1, 0), B(1,1) and C(0,1)] is mapped onto a parallelogram OA'B'C'.
   a) Obtain the coordinates of this parallelogram.
   b) Calculate the area of this parallelogram.
   c) Calculate the determinant of the transformation matrix.
   d) What conclusion did you reach?
Assessment 7

1. Let \( A = \begin{pmatrix} 2 & 5 & -1 \\ 3 & 2 & -1 \\ 1 & 3 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & 8 & -1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix} \).

Calculate:

a) \( A + B \)

b) \( 2B - A \)

c) \( A^2 \)

d) \( \text{adj}(B) \)

e) \( A^{-1} \) (the inverse of \( A \))

f) \( \det(A) \)

g) \( AB \)

h) \( (BA)^T \).

2. Solve the following system using the inverse matrix method:

\[-5x + 3y = 4 \]
\[3x - 5y = 10 \]

3. Solve the following system of equations using Cramer’s Rule:

\[2x + 3y - 2z = 1\]
\[x - 2y - 3z = -9\]
\[5x + 4y - 4z = 2\]

4. The wholesale (W), trade (T) and retail (R) prices for three products A, B and C

\[ W \begin{pmatrix} 8 & 11 & 16 \\ 7 & 10 & 14 \\ 11 & 15 & 21 \end{pmatrix} \]

are given by: \( P = B \begin{pmatrix} 8 & 11 & 16 \\ 7 & 10 & 14 \\ 11 & 15 & 21 \end{pmatrix} \)

Two stores I and II have numbers of stock given by:

\[ A \begin{pmatrix} 20 & 10 & 6 \end{pmatrix} \]
\[ \begin{pmatrix} 12 & 20 & 4 \end{pmatrix} \]

Obtain the matrix that represents the total wholesale, trade and retail values of all products in the two stores.

5. Using determinants find the equation of the plane through \( X(2, 9, -2) \), \( Y(0, -1, 7) \) and \( Z(2, 8, -2) \).

6. Using determinants find the area of the triangle that has vertices \((3, -2), (-4, 1)\) and \((8, 3)\).

7. Consider matrix transformation matrix \( M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \).

Use this matrix to transform the trapezium with coordinates \( A(2, 1), B(0, 1), C(1, 0) \) and \( D(2, 0) \) and interpret your result.
Answers to activities

Activity 7.1
1. a) 2 by 2  b) 2 by 2  
c) 3 by 3  d) 2 by 4

2. a) 9  b) 4  c) $n^2$

3. a) \[
\begin{pmatrix}
1 & 3 \\
2 & 4
\end{pmatrix}
\]
   b) \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
   c) \[
\begin{pmatrix}
3 & 2 & 4 \\
3 & 3 & -6 \\
5 & 5 & 0
\end{pmatrix}
\]
   d) \[
\begin{pmatrix}
1 & 6 \\
-2 & 2 \\
-3 & 0 \\
-4 & -1
\end{pmatrix}
\]

4. \[
\begin{pmatrix}
0 & -1 & -2 \\
1 & 0 & -1
\end{pmatrix}
\]

Activity 7.2
1. a) \[
\begin{pmatrix}
0 & 0 \\
9 & 2
\end{pmatrix}
\]
   b) \[
\begin{pmatrix}
4 & 2 \\
-3 & 2
\end{pmatrix}
\]
   c) \[
\begin{pmatrix}
5 & 1 \\
6 & 2 \\
5 & 17 \\
6 & 15
\end{pmatrix}
\]

2. \[
\begin{pmatrix}
1 & 1 \\
3 & 1 \\
30 & 18
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
5 & 8 \\
12 & -2
\end{pmatrix}
\]

4. $x = 0$ and $y = 5$

5. a) \[
\begin{pmatrix}
6 & 0 \\
-4 & 2
\end{pmatrix}
\]
   b) \[
\begin{pmatrix}
-12 & 3 \\
9 & -6
\end{pmatrix}
\]
   c) \[
\begin{pmatrix}
-6 & 3 \\
5 & -4
\end{pmatrix}
\]
   d) \[
\begin{pmatrix}
18 & -3 \\
-13 & 8
\end{pmatrix}
\]
Activity 7.3

1. a) \[
\begin{pmatrix}
14 & 0 \\
8 & -2
\end{pmatrix}
\]
b) \[
\begin{pmatrix}
2 & 1 \\
3 & 1
\end{pmatrix}
\]
c) \[
\begin{pmatrix}
5 & 11 \\
10 & 22
\end{pmatrix}
\]
d) \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

2. a) \[
\begin{pmatrix}
7 & 10 \\
15 & 22
\end{pmatrix}
\]
b) \[
\begin{pmatrix}
3 & -5 \\
5 & 8
\end{pmatrix}
\]
c) \[
\begin{pmatrix}
8 & 10 \\
20 & 18
\end{pmatrix}
\]
d) \[
\begin{pmatrix}
18 & 15 \\
40 & 48
\end{pmatrix}
\]
e) \[
\begin{pmatrix}
13 & 10 \\
40 & 53
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
AB \\
BA \\
AC \\
BC \\
DA \\
DC
\end{pmatrix} =
\begin{pmatrix}
0 & -5 \\
-3 & 25
\end{pmatrix}
\begin{pmatrix}
24 & -21 \\
-3 & 2
\end{pmatrix}
\begin{pmatrix}
25 & 5 & -10 \\
2 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
36 & 7 & -15 \\
-5 & -1 & 2
\end{pmatrix}
\begin{pmatrix}
0 & 5 \\
6 & 1 \\
21 & -24
\end{pmatrix}
\begin{pmatrix}
-1 & 8 \\
-1 & 6 \\
2 & -23
\end{pmatrix}
\begin{pmatrix}
4 & 1 & -1 \\
14 & 3 & -5 \\
27 & 5 & 12
\end{pmatrix}
\begin{pmatrix}
7 & 5 \\
10 & -12
\end{pmatrix}
\]

4. John: 86%, Bill: 54%, Jack: 29% and Mary: 59%

5. \[
\begin{pmatrix}
1780 & 0.96 & 0.85 & 261 \\
1780 & 1.16 & 0.85 & 261 \\
1280 & 0.76 & 0.85 & 132
\end{pmatrix}
\]
on Tuesday

Activity 7.4

1. \[
\begin{pmatrix}
2 & 3 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
-4 & 3 \\
3 & -2
\end{pmatrix} =
\begin{pmatrix}
(2)(-4) + (3)(3) & (2)(93) + (3)(-2) \\
(3)(-4) + (4)(3) & (3)(3) + (4)(-2)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

2. \[
\begin{align*}
2a + c &= 1 \\
3a + 2c &= 0 \\
a &= 2, \ c = -3
\end{align*}
\]

3. 2b + d = 0

3b + 2d = 1

b = -1, \ d = 2

3. Singular
Use quadratic, exponential, logarithmic and trigonometric functions and matrices

4. \( \mathbf{AB}^{-1} = \begin{pmatrix} \frac{3}{7} & 1 \\ \frac{7}{7} & \frac{7}{7} \\ \frac{15}{7} & -\frac{9}{7} \end{pmatrix} \quad \mathbf{B}^{-1} \mathbf{A} = \begin{pmatrix} -\frac{4}{7} & \frac{10}{7} \\ \frac{5}{7} & -\frac{2}{7} \end{pmatrix} \)

5. \( \mathbf{X} = \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \)

6. \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)

7. \( \mathbf{X}^{-1} = \frac{1}{20-18} \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 2.5 & 1.5 \\ 3 & 2 \end{pmatrix} \)

8. \( \frac{1}{k} \)

Activity 7.5

1. \( \begin{pmatrix} 2 & -3 \\ 5 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 22 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & 3 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 9 \\ 22 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \)

2. \( x = 4 \) and \( y = 7 \)

3. \( x = 3, y = 1 \) and \( z = -1 \)

4. a) \( \mathbf{MN} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

b) \( \mathbf{M}^{-1} = \frac{1}{2} \begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix} \)

c) \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 9 \end{pmatrix} \times \frac{1}{2} \begin{pmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -24 \\ 17 \\ 11 \end{pmatrix} \)

Activity 7.6

1. a) 24 \quad b) (-6) \quad c) 14

2. a) 30 \quad b) 21 \quad c) (-38)
3. \[
\begin{pmatrix}
3 & 2 & 4 \\
3 & 3 & -6 \\
5 & 5 & 0
\end{pmatrix}
\begin{pmatrix}
-0.5 & -2.5 & 1.5 \\
0.5 & 0.5 & -0.5 \\
0.5 & 3.5 & -1.5
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

a) 10  

b) 0.5  

c) 5  

d) 5

The determinant of a product is the product of determinants.

**Activity 7.7**

1. \[
\begin{pmatrix}
0 & 1 & -1 \\
1 & -2 & 2 \\
0 & 0 & 1
\end{pmatrix}
\]

2. a) 5  

b) \[
\begin{pmatrix}
-0.6 & 1 & 0.2 \\
-0.4 & 1 & 0.8 \\
0.8 & -1 & -0.6
\end{pmatrix}
\]

c) 25  

d) 0.2  

e) \[
\begin{vmatrix}
-3 & 5 & 1 \\
-2 & 5 & 4 \\
4 & -5 & -3
\end{vmatrix} = 25
\]

\[|A|^2 = |\text{adj}(A)| \text{ and } |A^{-1}| = \frac{1}{|A|}\]

3. a) \(x = 3, y = 1 \) and \(z = (-1)\)  

b) \(x = \frac{18}{7}, y = \frac{-2}{7} \) and \(z = (-1)\)

**Activity 7.8**

1. a) \(x = \frac{18}{11}, y = \frac{17}{11}\)  

b) \(x = 2, y = (-3)\)

2. a) \(x = 11, y = (-4) \) and \(z = 0\)  

b) \(x = (-2), y = 1 \) and \(z = 1\)

3. Solve: \(4v + 4x = 24\) and \(6v - 6x = 24\)  

Speed boat: 5 km/h. Speed river: 1 km/h

4. Solve \(8g + 4s = 104\) and \(6g + 6s = 84\)  

Price gold: $12/g. Price silver $2/g

5. 2 cleanser, 10 toner and 5 moisturiser

6. \(y = 1 - 2 \) or \(y = x^2 - 2x\)
7. \( i_1 = -2, i_2 = 1, i_3 = -1 \)

8. \( A = (-1), B = (-1) \) and \( C = 2 \)

**Activity 7.9**

1. \( M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \)

\( M^2 \) provides the number of two-step paths between the points.

2. a) \( A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \)

\( A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \)

b) \( A + A^2 + A^3 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix} \)

\( A \) is disconnected.

3. a) \( 256 \ 228 \ 340 \ 425 \ 256 \ 279 \ 396 \ 443 \) b) Flee

4. Rotation through 90° about O (quarter turn anticlockwise)

5. a) \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

b) Rotation through (–90°) about O (quarter turn clockwise)
Activity 7.10

1. \( K = \begin{bmatrix} 5 & 3 & 1 \\ 6 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix} = (1) \)

\[ p = -(5^2 + 3^2) = (-34) \]

\[ q = -((6)^2 + 2^2) = (-40) \]

\[ r = -((3)^2 + (4)^2) = (-25) \]

\[ D = \begin{bmatrix} -34 & 3 & 1 \\ -40 & 2 & 1 \\ -25 & 4 & 1 \end{bmatrix} = (3) \]

\[ F = \begin{bmatrix} 5 & 3 & -34 \\ 6 & 2 & -40 \\ 3 & 4 & -25 \end{bmatrix} = (28) \]

\[ E = \begin{bmatrix} 5 & -34 & 1 \\ 6 & -40 & 1 \\ 3 & -25 & 1 \end{bmatrix} = (3) \]

Hence the equation of the circle is \( x^2 + y^2 - 3x + 3y - 28 = 0 \)

The area of the triangle is \( \frac{1}{2} \times \text{absolute value of (1)} = 0.5 \)

2. Equation of circle is \( x^2 + y^2 + 2x - 4y - 8 = 0 \)

or \( (x + 1)^2 + (y - 2)^2 = 13 \)

Hence the in-centre is \((-1, 2)\)

3. 25

4. \(-3x + 9y - 6z = 0\)

5. \(4x - 2y + z - 8 = 0\)

6. a) \(O'(0, 0), A'(2, 1), B'(3, 4)\) and \(C'(1, 3)\)

b) 5

c) 5

d) If \( T \) is the transformation matrix than the area of the image has become larger by a factor of \( \det(T) \), ignoring the sign of \( \det(T) \).
Use quadratic, exponential, logarithmic and trigonometric functions and matrices.
USE QUADRATIC, EXPONENTIAL, LOGARITHMIC AND TRIGONOMETRIC FUNCTIONS AND MATRICES
EDX140

Learner’s Guide

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This EDX140 unit covers various ‘pre-calculus’ mathematical techniques relating to algebra, trigonometry and matrix algebra. These techniques may be described as essential knowledge in order to study technology oriented subjects at Diploma level such as fluid dynamics and statics. Students will need to be thoroughly familiar with the material that will be covered if they wish to study an engineering related field beyond Diploma level in which calculus is essential.

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